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I. Technical Report Summary

Two major advances have been made in the research on synthesis of long-period S waves at distances less than 40°: the incorporation of inelastic attenuation and the development of a code to include a representation of the source function. The large effects on the waveforms of rapid variations of velocity in the upper mantle can be calculated and separated from the effects of anelasticity.

Examples of synthetic seismograms are included to illustrate the effects of attenuation on the displacement waveforms, as well as the effects of the response of a typical long-period seismograph system.

The application of Green's Function techniques to elastodynamics has led to methods for treating a variety of problems in wave propagation and earthquake source representations. The complete theory is presented in this report. Wave propagation is a realistic, layered earth, generalized by a nonlinear source and the dynamic field due to stress relaxation around a geometrically general, growing inclusion (an earthquake source) in a spatially heterogeneous initial stress field are two of the significant problems that have been treated. Computations based on the equations derived here are currently being carried out.

There is evidence that specific anelastic attenuation, expressed as Q, is frequency dependent. Q models of the Earth based on free oscillations, surface waves and body waves show that in the broad frequency band covered by the input data, Q increases with frequency. Frequency-dependent Q is modelled by a relaxation process with a range of relaxation times. An investigation of the relaxation time characterizing the high frequency corner of the absorption based was carried out using the data from 21

shallow earthquakes, 1 at intermediate depth, and 4 deep ones.

The criteria used was that the Q-corrected spectrum show decay at high frequencies at a rate in the range f^{-2} to f^{-3} . The depth dependence of the resulting relaxation times and corresponding values of T/Q was examined. A mixed effect of depth and frequency dependence of P-wave attenuation was found. The important conclusions is that the P and S attenuation data can only be reconciled by including a bulk loss mechanism, in addition to a shear loss mechanism. Although the results are not unique, this suggests that the bulk loss mechanism is operative in the upper mantle, perhaps within the asthenosphere.

II. Synthesis of Long Period Body Waves in an Anelastic Earth

V.F. Cormier

Sunthesis of long period SH body waves has been completed in the PEM-C (Figure 2.1) earth model of Dziewonski et al. (1975). This earth model in conjunction with a simple frequency independent Q model was chosen as a starting model for inversion for the upper mantle structure of western North America. Synthetic modeling of waveforms at distances less than 40 degrees must be undertaken to separate the large effects of rapid velocity variation in the upper mantle from the effects of anelasticity. The incorporation of attenuation in the analysis through a frequency dependent complex velocity profile has been taken as complementary check of spectral studies of upper mantle attenuation structure.

Seismograms have been synthesized using the method described by Cormier and Richards (1977), in which the Fourier-transformed displacement is evaluated by the evaluation of an integral over short paths in the complex ray parameter plane. Langer's approximation to the radial eigenfunctions in the integrand of reflection-transmission coefficients corrects for the effect of velocity gradients and boundary curvature of inhomogeneous spherical layers. In the last funding period a significant improvement in the computation speed of synthetics has been achieved by using the three point integration formula described by Jeffreys and Jeffreys (1956) for the evaluation of the quantity

$$\tau = \int_{r_p}^{r} \left(\frac{1}{\beta^2} - \frac{p^2}{r^2} \right) dr$$

in inhomogeneous layers. This quantity is required to evaluate the Langer

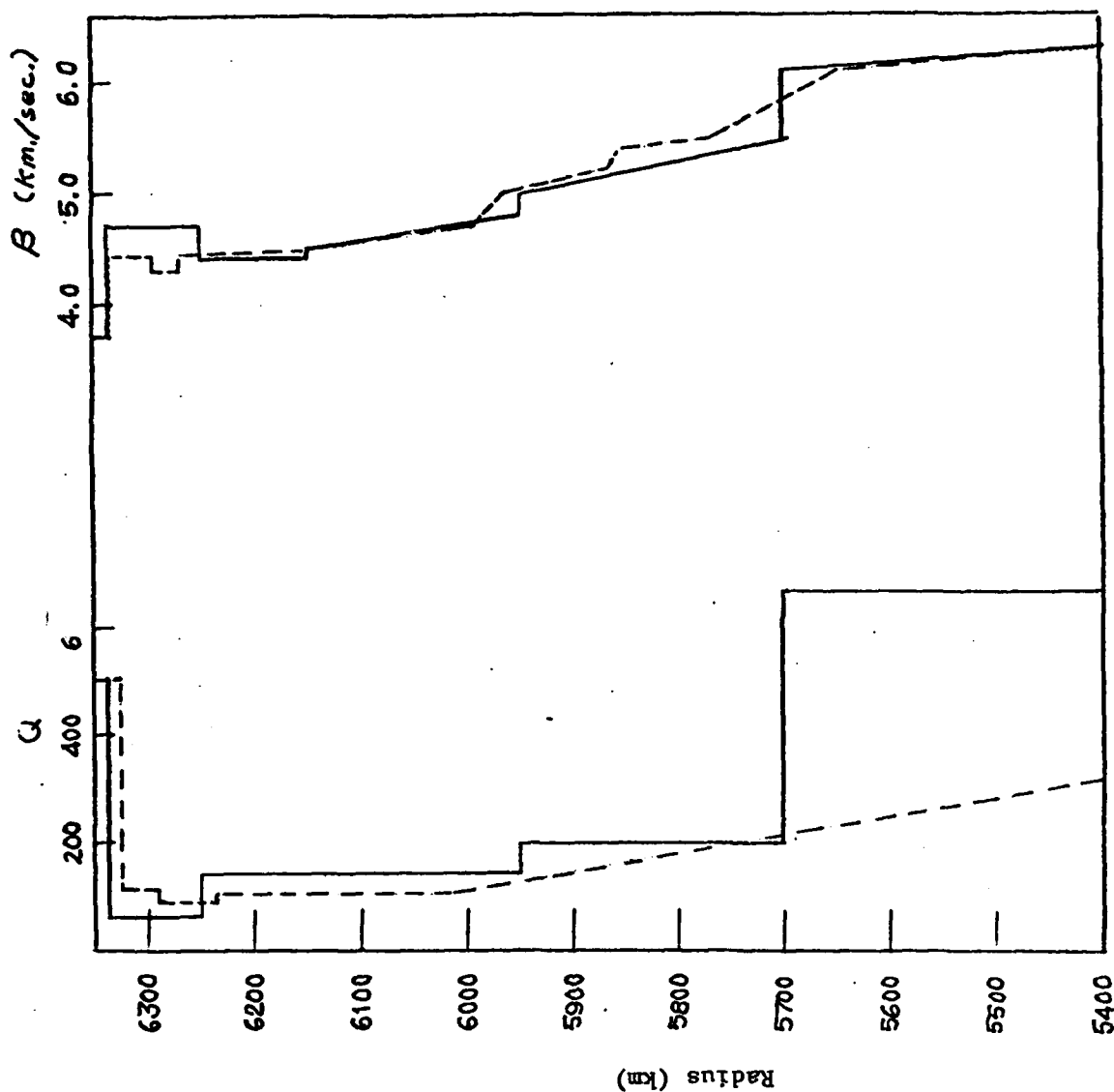Figure 2.1: S velocity predicted by PEM-C in the upper mantle and a simple model for Q_β used in calculations. A recent Q_β model by Hart (1977) and S velocity model SHR-14 by Helmberger and Engen (1974) are dashed.

approximation (Richards, 1976). By use of the three point formula, the computation of synthetic seismograms for direct SH body waves in an attenuative-dispersive earth model with two upper mantle $T-\Delta$ triplications requires only 10 minutes execution time on a CDC-6600 at 2 degree intervals from 20 to 34 degrees.

Multiple reflections along the underside of discontinuous velocity increases in the upper-mantle of PEM-C are included by use of generalized reflection coefficients and integration paths of the type described in Cormier and Richards (1977). For example, the reflection coefficient at critical incidence is taken as the functions

$$R_{22} \frac{\sigma_2^{(2)}}{\sigma_2^{(1)}} \quad , \quad S \frac{\sigma_2^{(2)}}{\sigma_2^{(1)}} \quad , \quad T_{21} T_{12} \frac{\sigma_2^{(2)}}{\sigma_2^{(2)}} \frac{\sigma_1^{(1)}}{\sigma_1^{(2)}} \bigg/ \left(1 - R_{11} \frac{\sigma_1^{(1)}}{\sigma_1^{(2)}} \right)$$

along the integration paths illustrated in Figure 2.2, where R_{22} is the reflection coefficient of SH from the top of the boundary, R_{11} the reflection coefficient from the bottom, T_{21} , T_{21} transmission coefficients from the top or bottom respectively, $\sigma_2^{(1)}$, $\sigma_2^{(2)}$ radial eigenfunctions for (1) up or (2) downgoing SH waves in the upper medium, $\sigma_1^{(1)}$, $\sigma_1^{(2)}$ radial eigenfunctions in the lower medium. These coefficients are related by the equation

$$S = R_{22} \frac{\sigma_2^{(2)}}{\sigma_2^{(1)}} + T_{21} T_{12} \frac{\sigma_2^{(2)}}{\sigma_2^{(1)}} \frac{\sigma_1^{(1)}}{\sigma_1^{(2)}} \bigg/ \left(1 - R_{11} \frac{\sigma_1^{(1)}}{\sigma_1^{(2)}} \right) \qquad (2.1)$$

At distances much greater than critical a reflection coefficient is constructed from the functions above for use along the integration path illustrated
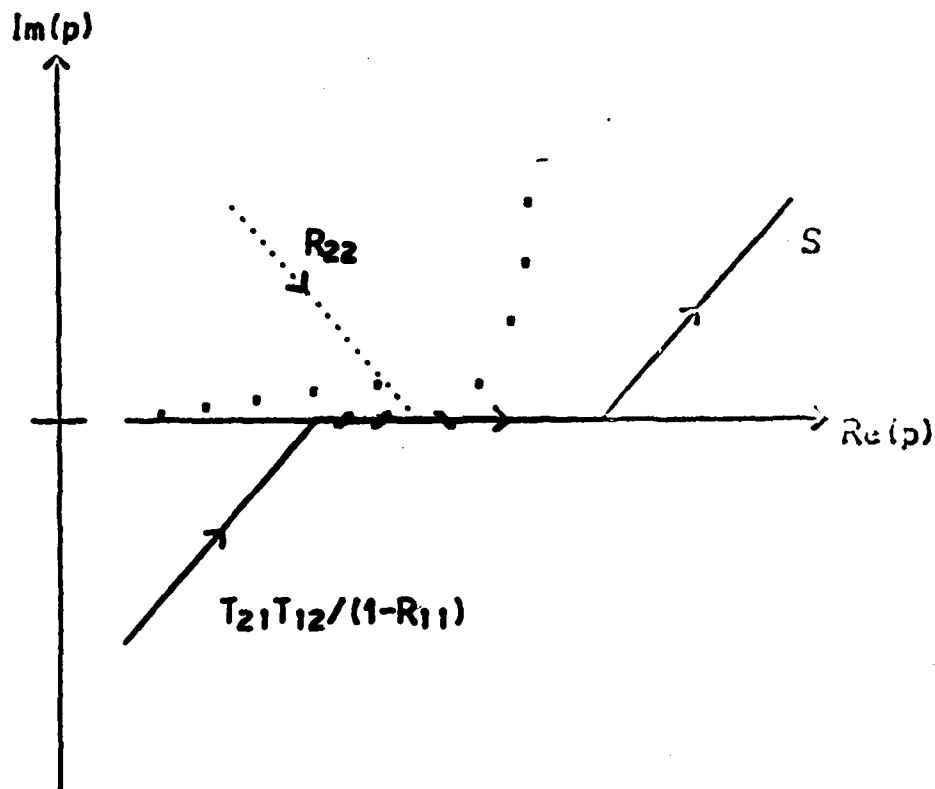
PATH FOR CRITICAL DISTANCE

Figure 2.2: Integration paths near a cusp at critical incidence. The coefficients are taken along the segments as shown. (Ratios of radial eigenfunctions $\sigma^{(1)}/\sigma^{(2)}$ are omitted in the figure.

in Figure 2.3, where

$$S - R_{22} \frac{\sigma_2^{(2)}}{\sigma_2^{(1)}} - T_{21} T_{12} \frac{\sigma_2^{(2)}}{\sigma_2^{(1)}} \frac{\sigma_1^{(1)}}{\sigma_1^{(2)}} \sum_{n=0}^{N} \left[R_{11} \frac{\sigma_1^{(1)}}{\sigma_1^{(2)}} \right]^n =$$

$$T_{21} T_{12} \frac{\sigma_2^{(2)}}{\sigma_2^{(1)}} \frac{\sigma_1^{(1)}}{\sigma_1^{(2)}} \sum_{n=N+1}^{\infty} \left[R_{11} \frac{\sigma_1^{(2)}}{\sigma_1^{(2)}} \right]^n \qquad (2.2)$$

The N waves constituting the direct transmitted and N-1 multiples
are included in a separate ray parameter integral when they are well separated
in arrival time from multiples described by eq. (2.2).

By proper choice of combinations of these functions and integration
paths, the effect of all the interfering multiples in two upper mantle
triplications can be included in the evaluation of a single integral over
ray parameter in the distance range near 20°.

Figure 2.4a-b shows the results of the synthesis for the response
of the PEM-C model to a SH source for the direct body wave. Comparison
of the results with (2.4b) and without (2.4a) attenuation demonstrate
that the removal of high frequencies by attenuation obscures details
in the waveform due to multiple arrivals from first order discontinuities
in the earth model. The difference in model response remains visible even
after inclusion of the transfer function of a long period seismograph
(Figures 2.5a- .5b). These results are consistent with those reported by
Kennet (1975) for inclusion of attenuation in the reflectivity method of
seismogram synthesis.
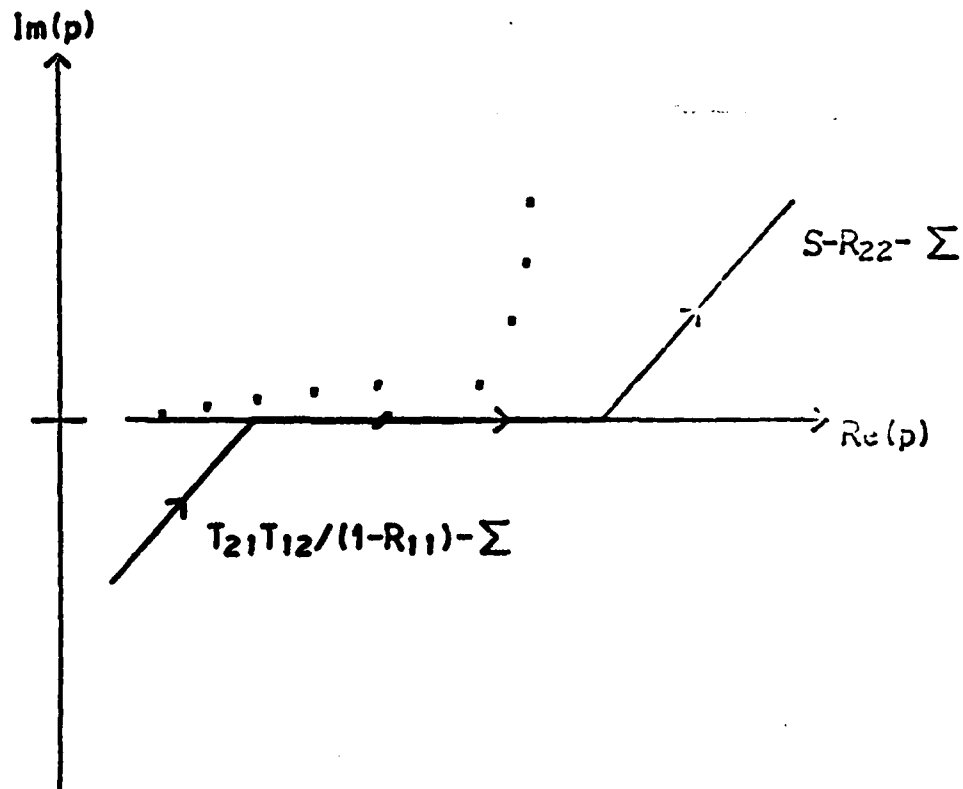
PATH FOR HEAD WAVE

Figure 2.3: Integration path for an interference head wave.
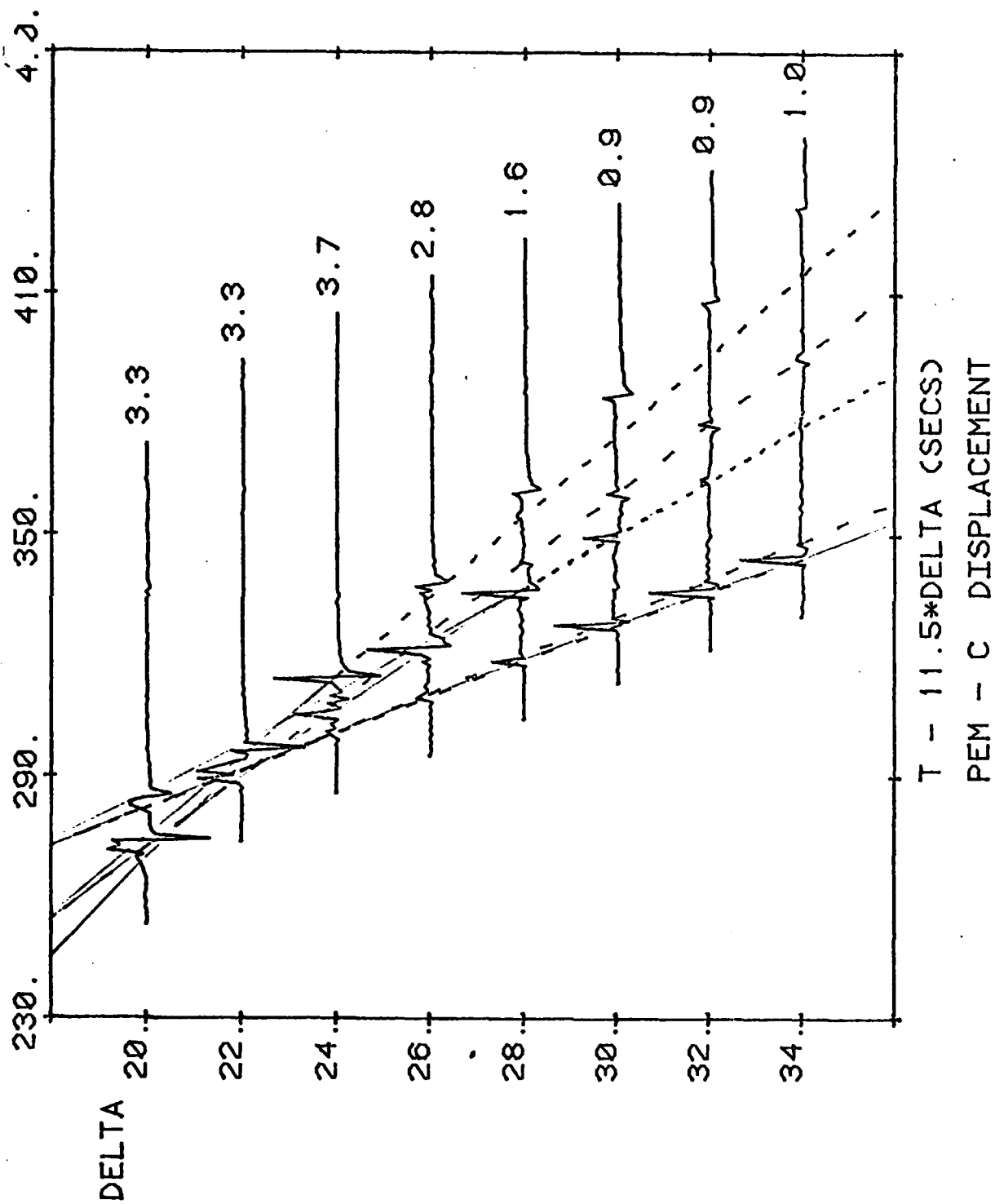
Figure 2.4a: PEM-C response for an SH source, no attenuation i ...
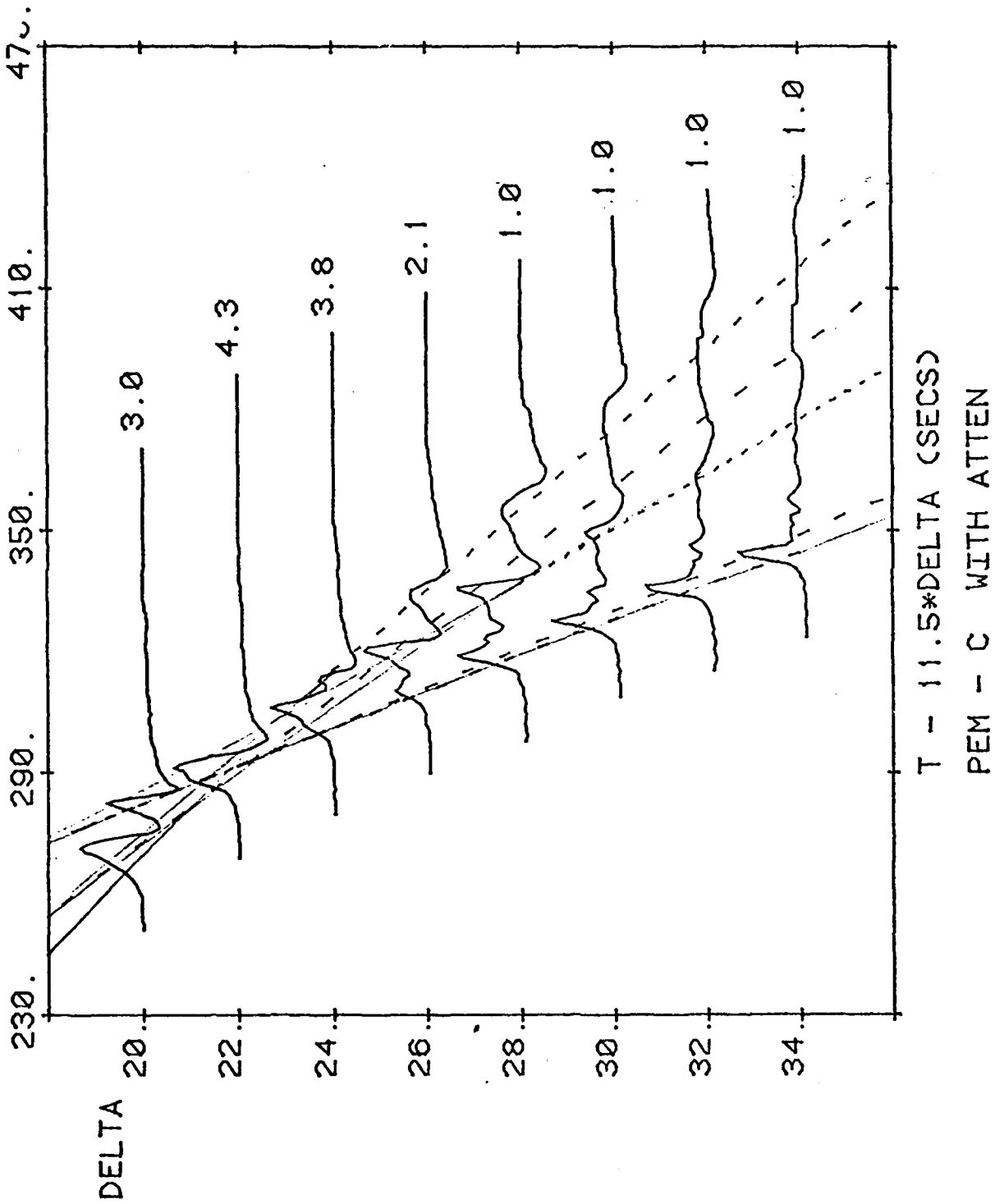
Figure 2.4b: PEM-C response for SH source, simple attenuation model of Figure 1 assumed.

470.

410.

350.

290.

230.

5.6

5.1

3.3

2.5

1.1

0.8

0.9

1.0

DELTA

20.

22.

24.

26.

28.

30.

32.

34.

T - 11.5*DELTA (SECS)

PEM - C DISPL * INSTRU
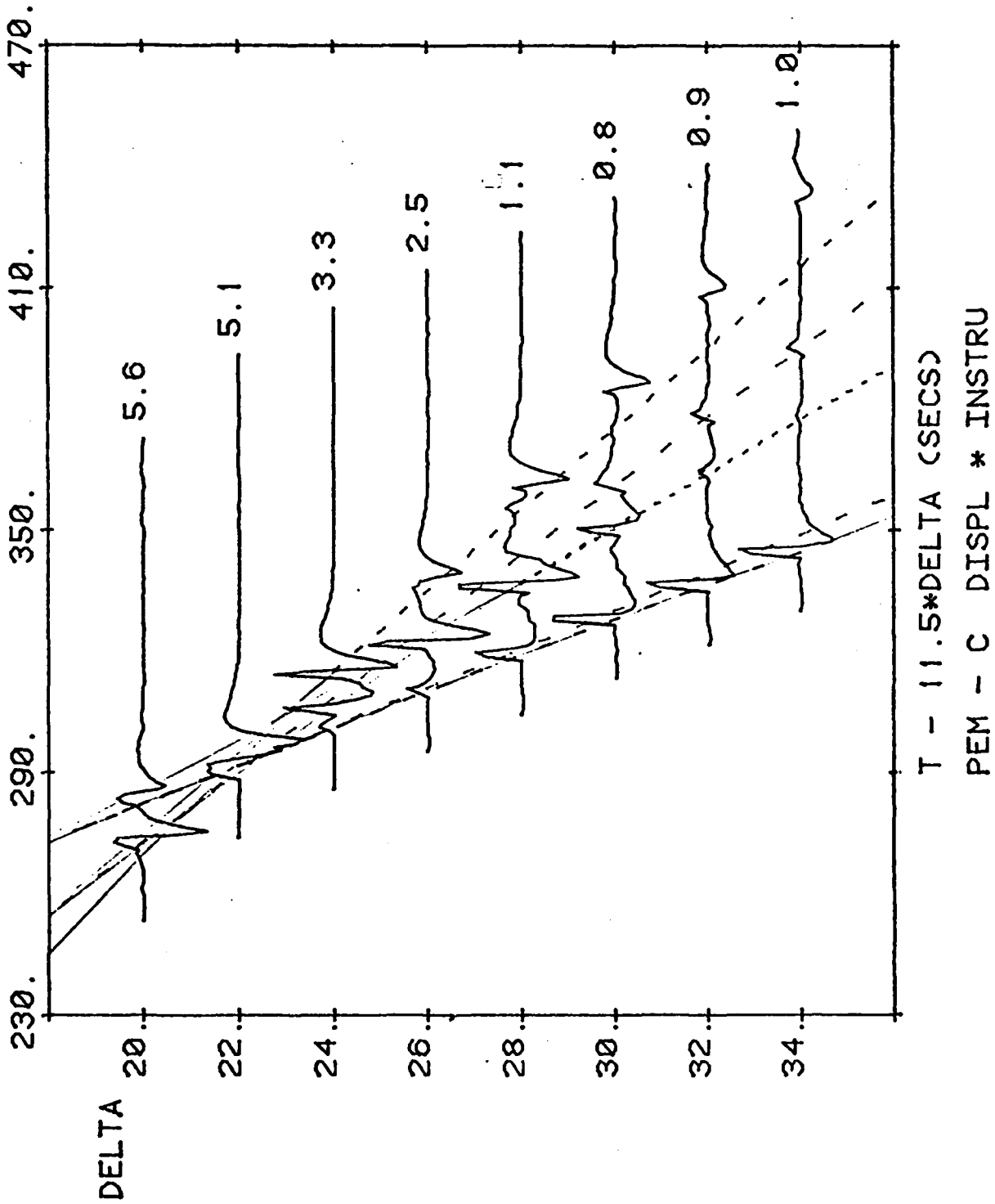
Figure 2.5a: PEM-C response with no attenuation convolved with 15-100 WWSSN response.
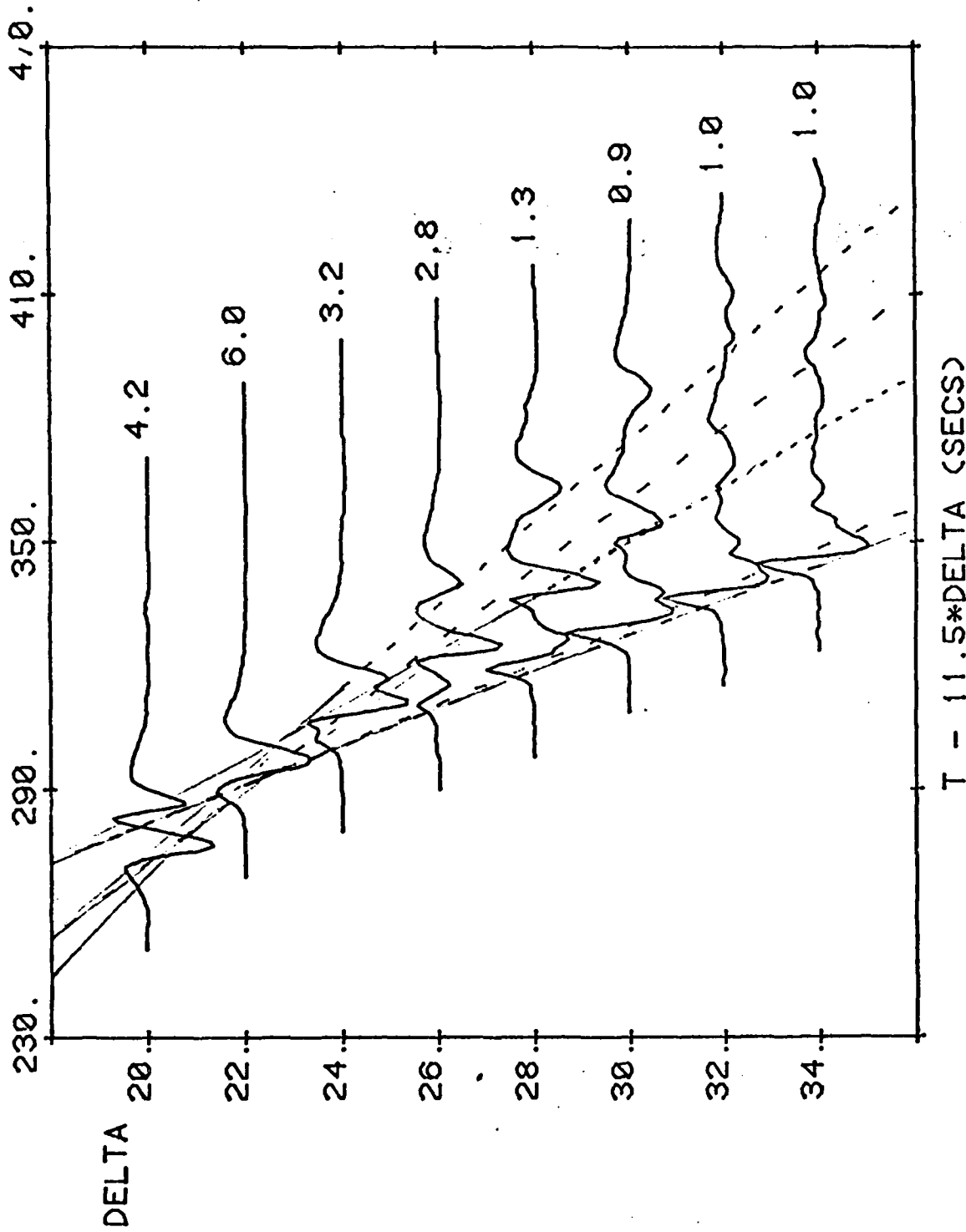
Figure 2.5b: PEM-C response with attenuation convolved with 15-100 WWSSN response.

Figure 2.6 compares synthetics generated by convolving the model

response with a source function determined for the Borrego earthquake

(Helmberger and Engen, 1974) with observed seismograms of the earthquake.

The source function includes near source multiples and attenuation through the

Carpenter (1966) Q operator. The model response for no attenuation

consequently was the one convolved. Future research will compare wave-

forms predicted with the Carpenter Q operator with those determined by

including attenuation in the frequency domain. The latter approach has the

advantage of more easily handling complicated frequency dependence of Q,

but requires slightly longer computation time.

Analysis of Figure 2.6 indicates that of the later arrivals predicted

by the PEM-C model, the one due to reflection from the discontinuity at

671 km is not greatly in disagreement with the observed seismograms. The arrival

forming the latest and largest trough in the synthetics corresponds to a wave

diffracted along the top of the lower boundary of the low velocity zone of

PEM-C. Revision of the PEM-C model to agree with waveform (Figure 2.6)

travel times (Figure 2.7) can be simply accomplished

by modifying the structure of the low velocity zone. Unfortunately wave-

forms cannot sufficiently constrain deeper structure because a long period

recording of a sufficiently large earthquake cannot resolve the features

of waveform due to interfering triplications near 20 degrees. Attenuation

makes this resolution more difficult. One must rely on carefully matching

amplitudes near 20° and matching high quality waveforms at distances greater

than 30 degrees uncontaminated by PL waves.

In the next research period SH synthesis will be completed in earth

models satisfying travel-time, amplitude, and waveform data for western
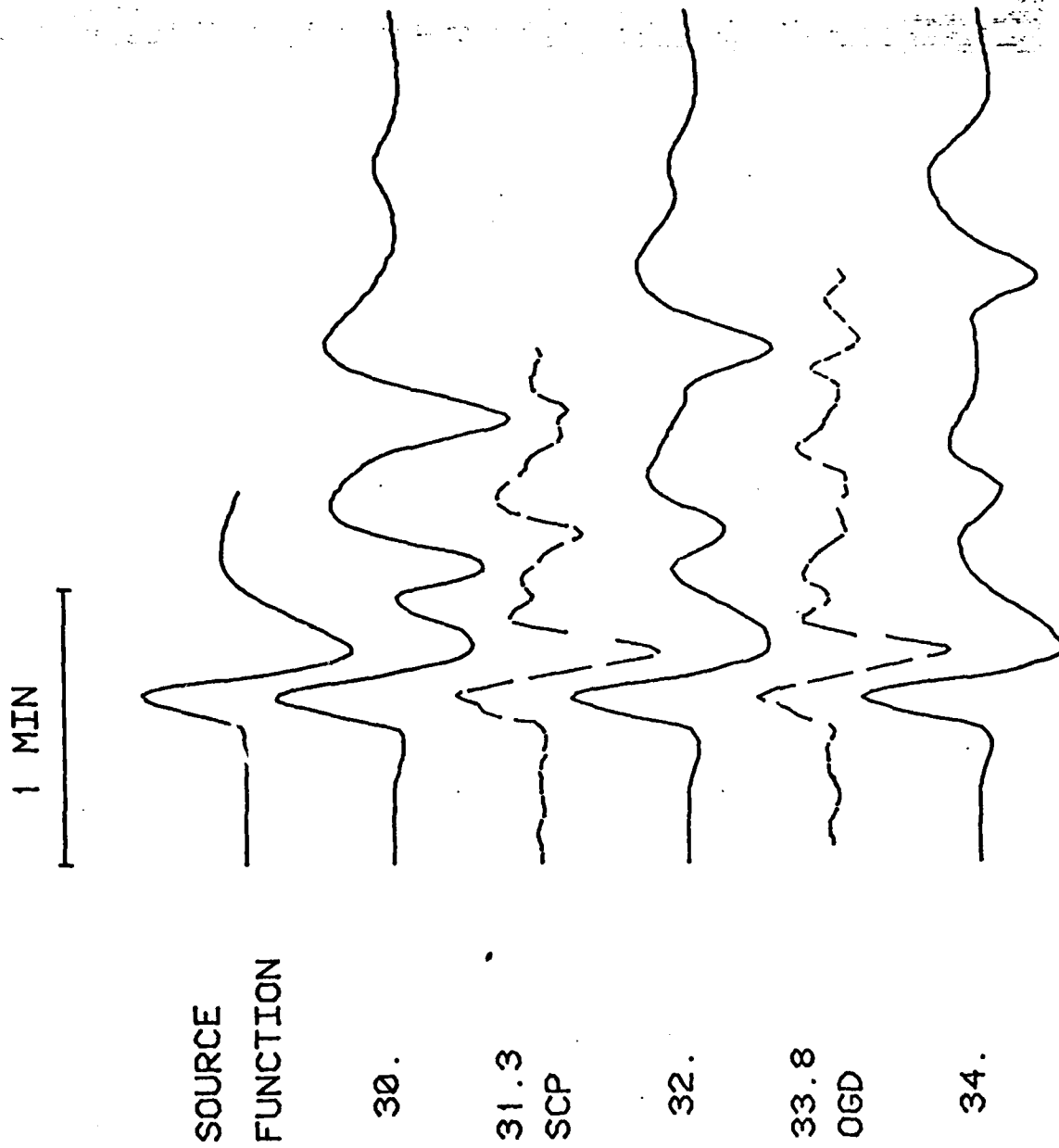
Figure 2.6: Synthetics (30, 32, 340) generated by convolving model response with Burrego Mountain earthquake (April 9, 1966) that given by Helmberger and Engen (1966); data at WWSSN stations SCP and OGD.
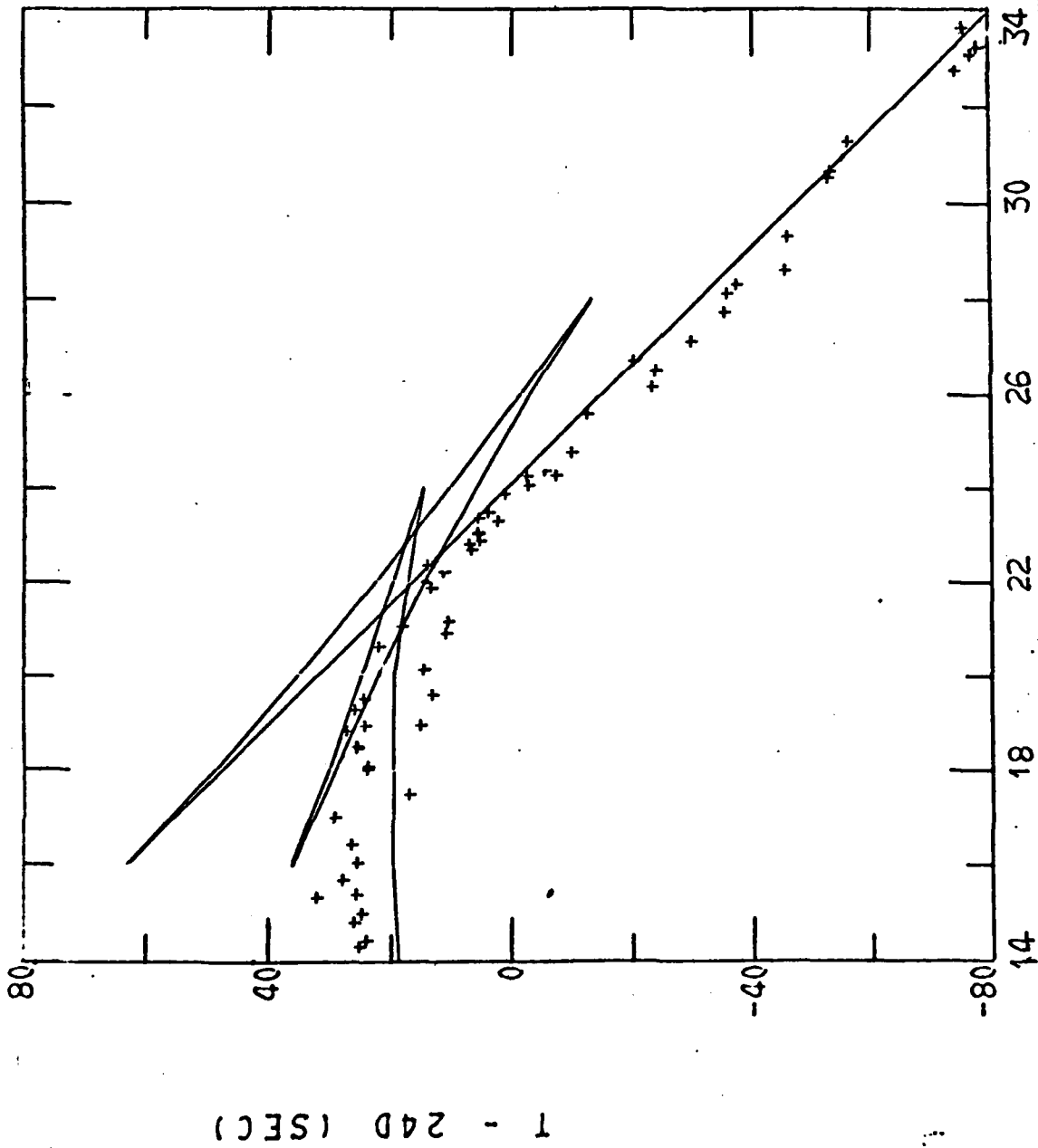
Figure 2.7: Reduced travel time curve in PEM-C compared with observed travel times (+ - S).

North America. A range of upper mantle models for S velocity and Q will be suggested. The validity of including attenuation in the time domain via the Carpenter Q operator will be investigated.

8

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Kennet, B.L.N. (1975). The effects of attenuation on seismograms. Bull. Seism. Soc. Am., 65, 1693-1652.

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III. Anelastic Properties of the Earth Inferred from Earthquake Spectra:
Investigations of Frequency Dependent Q Models

G.M. Lundquist

Introduction

Anelastic attenuation of seismic waves is parameterized by the specific

quality factor, Q , defined by

$$Q^{-1} = \frac{1}{2\pi} \frac{\Delta E}{E} \qquad (3.1)$$

where ΔE is the energy dissipated in a cycle whose average energy is

E . In a layered medium, the wave is attenuated proportional to a weighted

average of the Q^{-1} in each layer, where the weight is given by the travel

time.

$$Q_{Eff}^{-1} = \frac{\sum\limits_{i} \frac{t_i}{Q_i}}{\sum\limits_{i} t_i} \qquad (3.2)$$

where t_i is the travel time in each layer. The functional dependence of

the absorption becomes apparent in the definition,

$$\text{TOTAL ANELASTIC ATTENUATION} = e^{-\frac{\omega T}{2Q_{Eff}}} \qquad (3.3)$$

where T is the total travel time. For convenience, we will note the

ratio T/Q_{Eff} by t^* with subscript α or β for P or S waves respectively.

The minimum functional dependence for t^* must be on velocity and Q structure

along the ray path.

If an additional dependence upon frequency is to be found, then the

velocity and Q structures must either be known or somehow eliminated from

the problem. As before (Lundquist, 1975) we assume a multiplicative

separability between the depth dependence and frequency dependence of the

form

$$t^*(r, \omega) = t^*(r) \, R(\omega) \qquad\qquad (3.4)$$

where r is radius in a spherically symmetric earth model and $R(\omega)$ carries all of the frequency dependence. Equation (3.4) demonstrates explicitly the nonuniqueness inherent in determining $R(\omega)$ when t^* is not known. Unfortunately, since t^* appears in an exponent, ratioing techniques will not isolate $R(\omega)$. That is, the base t^* functions must be estimated independently.

Toward this end, a review has been made of published velocity and Q models. Rather than finding a consensus among Q models, a frequency dependence is noted depending upon the data set used. In gereral, Q seems to be an increasing function of the frequency of the data set. To determine whether this frequency dependence may be related to $R(\omega)$ as determined from body-wave spectra, preliminary attempts are made to model the differences between published Q structure as a function of a frequency dependent Q, with quite good results.

Published Models.

Velocity models are quite well constrained by both body-wave and free-oscillation inversions. The slight (1%) base-line shift in theoretical travel times was shown to be a result of ignoring anelasticity in the free oscillation studies. Hart et al. (1977) corrected the entire set of known spheroidal and toroidal modes for attenuation and reduced the base-line shift from the Jeffreys-Bullen tables to less than a second. Since the resulting velocity model, QM2, agrees with both body-wave and free-oscillation data, it has been adopted for use in the present study.

Q models, on the other hand, are neither as well constrained nor as consistent. The range of models is shown in Figure 3.1. Model SL1 was
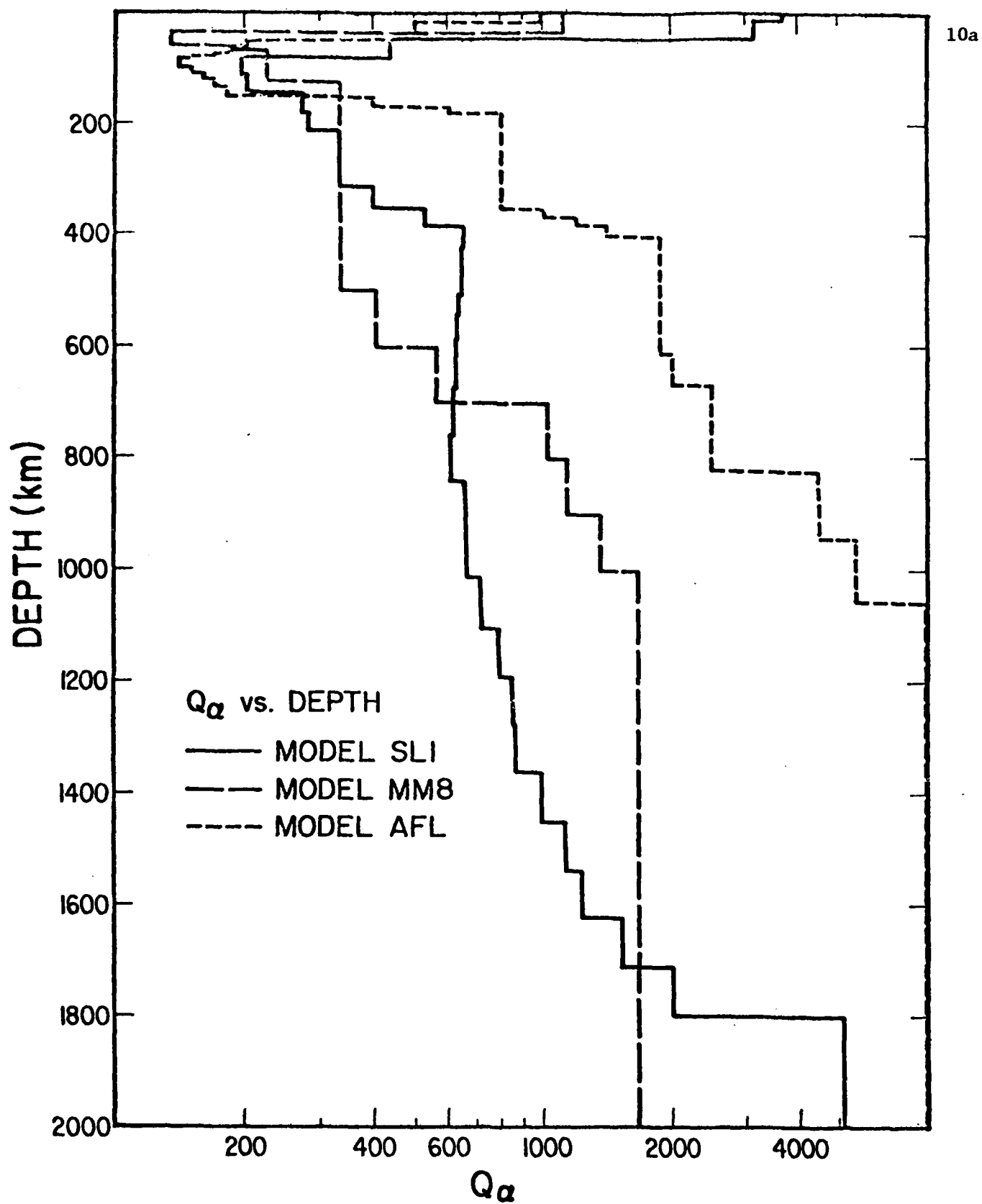
Figure 3.1: Q_α vs. depth. Models SL1 and MM8 were derived from free oscillations and surface waves, respectively. Model AFL was derived from body waves.

derived from free-oscillation data (Anderson and Hart, 1977); MM8 was

derived from surface-waves (Anderson and Archambeau, 1964); and AFL was

derived from P body waves (Archambeau et al., 1969). The frequencies in

the data sets appropriate to each model are given in Table 3.1. Note

that while the free-oscillation and surface-wave models overlap in frequency,

the body-wave model is taken from waves in a completely independent frequency

range.

TABLE 3.1

Model	Data Frequencies
SL1	.0003-.015 Hz
MM8	.0025-.02 Hz
AFL	2-5 Hz

The variation in Q models was examined as a function of t^* .
Figure 3.2 shows t^* vs epicentral distance for a shallow focus earthquake
for each of the models of Figure 3.1. Model SL1 included both Q_α and
Q_β . For the other two models, Q_β was obtained by $Q_\alpha = 2.35 Q_\beta$.
When combined with velocity model QM2, this Q ratio gives $t_\alpha^*/t_\beta^* \simeq 4.3$,
while the ratio for SL1 is $t_\alpha^*/t_\beta^* \simeq 4.55$. There is very little change in
the shape of t^* vs distance as a function of source depth for any particular
model.

In both Figures 3.1 and 3.2, the low-frequency models overlap, as might
be expected since they deal with data in the same frequency range. t^*(AFL) ,
however, is only half of the values for t^*(MM8) because Q(AFL) is con-
sistently higher than that for the other models. If both SL1 and AFL are
correct, then Q must increase with frequency.
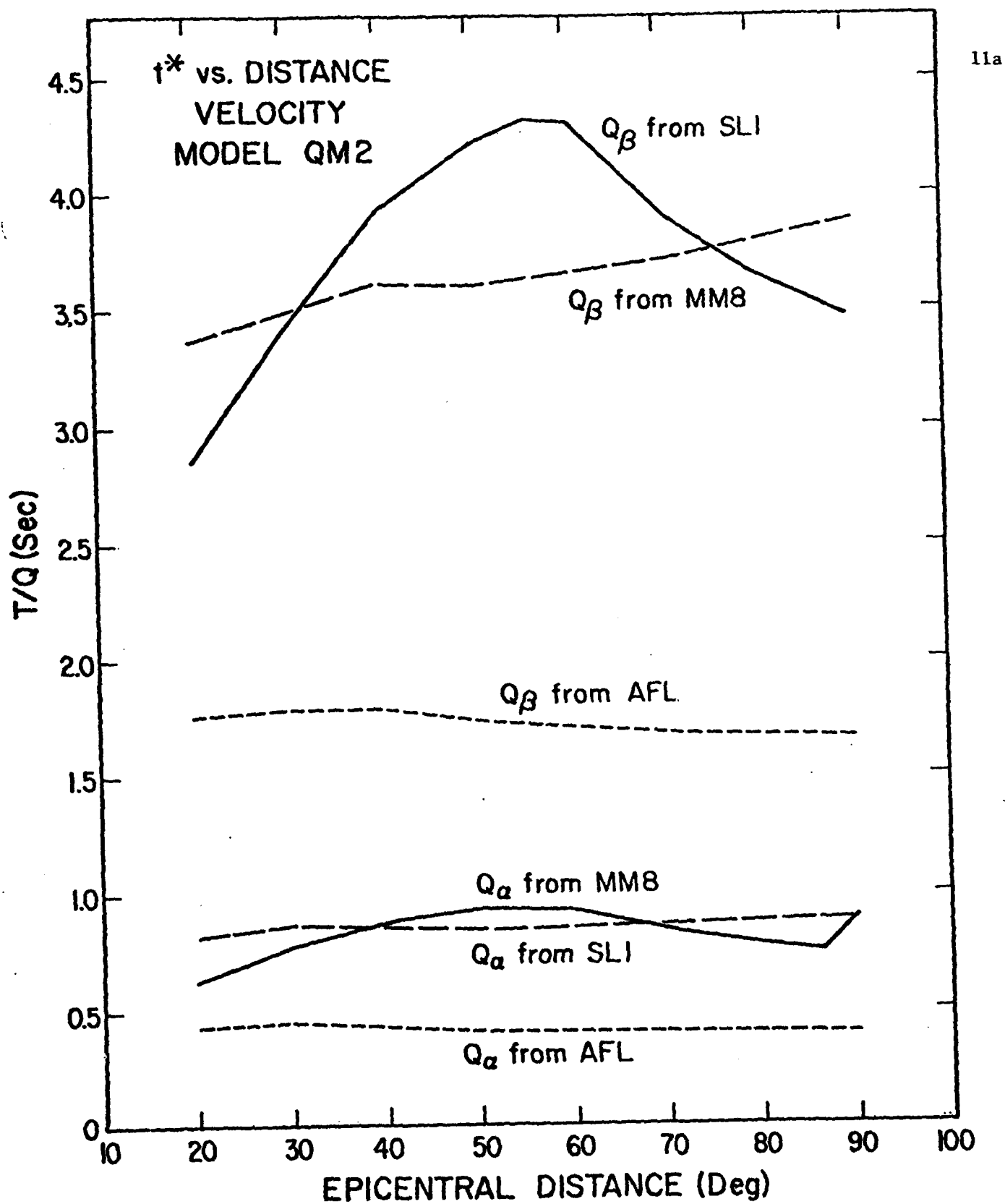
Figure 3.2: T/Q vs. Distance. The velocity model used was QM2. Note that T/Q(AFL) is well below T/Q for the low frequency models.

Figure 3.3 shows the variation of t^* vs hypocentral depth as a function of percent of surface focus value. Low frequency waves apparently miss about 23% of the total attenuation if the source is at 600 km depth. High frequency waves, on the other hand, apparently miss more than 40% for the same change in source depth, and the difference seems to occur over the depth range of the asthenosphere. This difference may also be reconciled simply as a function of frequency if low and high frequency waves see different changes in apparent Q as a function of source depth.

Q(f) Model.

Before going further, it is appropriate to briefly review the form of the frequency dependence of Q used here. The function, $R(\omega)$, in the attenuation exponent of equation (3.3) generates an absorption band from a constant T/Q_{eff} . The absorption band is constructed theoretically as the superposition of specific absorption mechanisms each of which is modelled by a standard linear elastic solid. Each separate mechanism attenuates according to

$$Q^{-1} = C \frac{\omega\tau}{1 + \omega^2\tau^2}$$

where C is a constant depending upon the elastic parameters of the medium and τ is the medium relaxation time (see Mascn, 1958). A distribution of relaxation times of the form

$$D(\tau) = \begin{cases} 1/\tau \ , \ \tau_1 > \tau > \tau_2 \\ 0 \ , \ \tau \geq \tau_1 \ \cdot \ \tau \leq \tau_2 \end{cases}$$

defines an absorption band as (Liu et al., 1976)
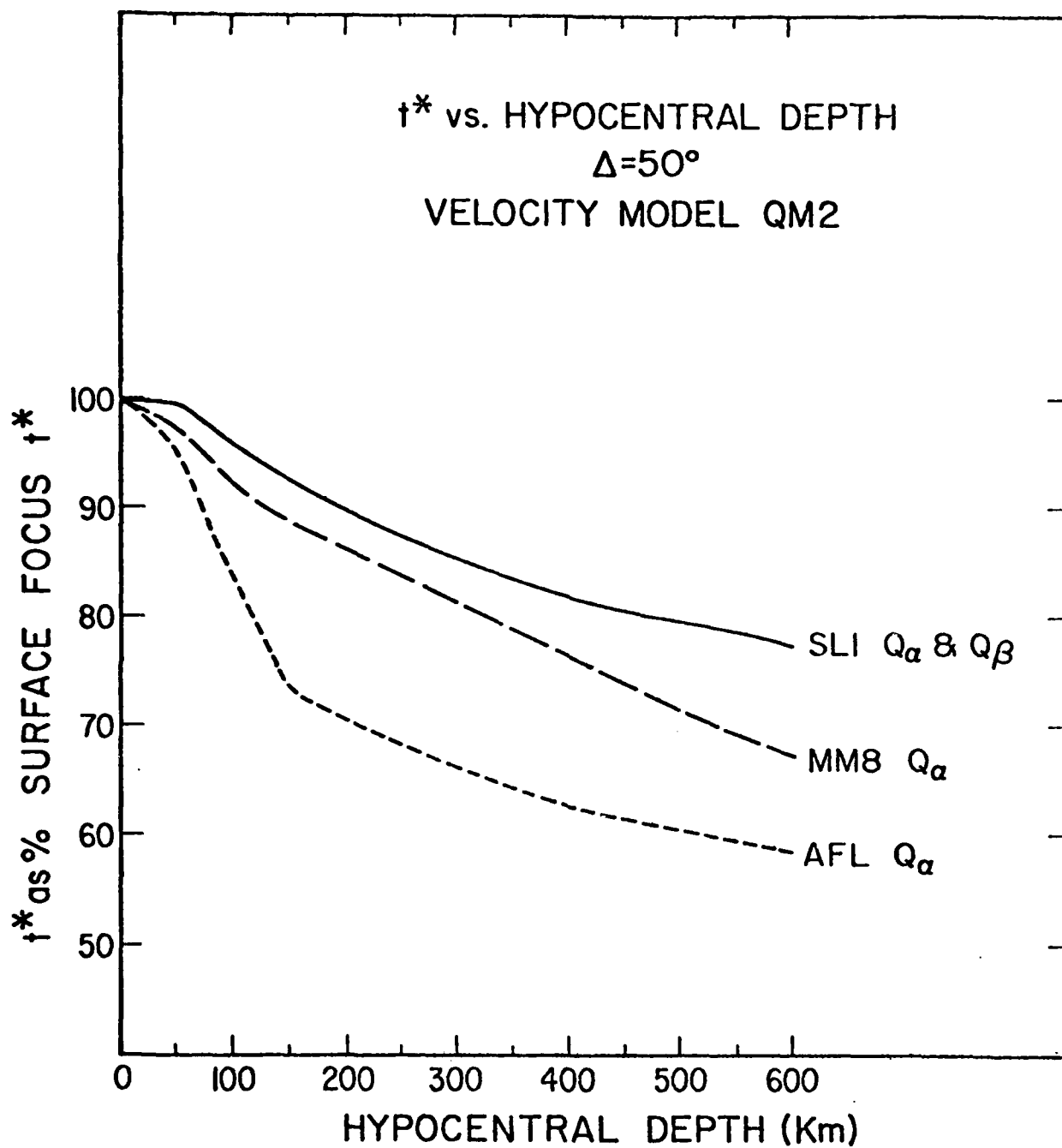
Figure 3.3: T/Q vs. Source Depth. The velocity model used was QM2. Note the variation as a function of the frequency in the data set used to generate the Q model.

$$Q^{-1} = C \frac{2}{\pi} \tan^{-1} \left\{ \frac{\omega(\tau_1 - \tau_2)}{1 + \omega^2 \tau_1 \tau_2} \right\} \qquad (3.5)$$

The parameter of interest in (3.5) is τ_2, the high-frequency half-power point of the absorption band. The other corner period is arbitrarily placed well beyond the longest periods of interest in body-wave studies at $\tau_1 = 2000$ sec. Thus the only frequency dependence introduced in this work is a roll-off in absorption toward high frequencies which may be varied by changing τ_2.

The point of introducing a frequency dependent Q is to recover observed seismic source spectra from seismograms. Theoretical source spectra all show amplitude decays toward high frequencies of the order ω^{-2} to ω^{-3}. The slope of an observed spectrum, however, is controlled much more by the anelastic attenuation along the travel path than by initial source properties. If a Q correction function is appropriate, it should give back an observed spectrum with slopes in the theoretical range. Thus $R(\omega)$ is used in the attenuation correction as a decay slope modification. The results of that modification will be presented in the next section.

Body Wave Spectra.

To date, 21 shallow earthquakes, 4 deep and 1 intermediate depth event have been studied. Body-wave spectra were computed from digitized seismograms using an autocorrelation technique which smoothed the individual spectra. Additional smoothing was obtained by averaging several stations for each wave type of each event studied. The result was usually (but not always) a noise-free spectrum whose high frequency decay slope could be uniquely fit by a single line segment.

An average spectrum without an anelastic attenuation correction was prepared for each event, under the assumption that T/Q_{eff} is nearly constant with epicentral distance. Then each average spectrum was corrected for attenuation with a range of τ_2. Initially, the base t^* function was assumed to be $t^*_\alpha = 1.0$ and $t^*_\beta = 4.0$; and the variation of t^* with hypocentral depth was chosen to be that of model AFL. After the review of published models was begun, this choice of t^* was seen to mix high and low frequency models. Thus the data set was reprocessed with t^* essentially following model SL1. For that test, $t^*_\alpha = 1.0$ and $t^*_\beta = 4.5$ for shallow events, and the SL1 depth dependence from Figure 3.3 was applied for deep and intermediate events.

The results of the tests are given in Tables 3.2 and 3.3 for shallow and deep events, respectively. The values $\tau_2 = \infty$ for ω^{-3} slopes of deep P waves result from the fact that the uncorrected station spectra already have slopes less than ω^{-3}. Since the Q correction can only decrease the decay rate, an ω^{-3} decay rate cannot be obtained for any τ_2. This may imply a significant difference between deep and shallow source mechanisms, or it may imply that the assumption of ω^{-3} slope is incorrect.

In Tables 3.2 and 3.3, the values of τ_2 in parentheses are the results of reprocessing with t^* defined by model SL1. Note that the result of increasing t^* is to increase the applied attenuation correction and raise the slope. Thus a larger τ_2 is required to get back the same slope for increased t^*. Though the difference is negligible for shallow S waves, the difference between t^* from models AFL and SL1 (or MM8) as a function of source depth is significant. If any physical understanding

TABLE 3.2

EVENT LIST SHALLOW EARTHQUAKES

Date	Depth	$\tau_{2\,p}(\omega^{-2})$	$\tau_{2\,p}(\omega^{-3})$	$\tau_{2\,s}(\omega^{-2})$	$\tau_{2\,s}(\omega^{-3})$
72/08/30	33	.09	.17	.17(.18)	.24(.24)
72/08/30	33	.11	.18	.16(.16)	.24(.24)
72/08/09	15	.06	.12		
72/04/09	33	.08	.16	.17(.18)	.26(.26)
71/05/22	33	.08	.18	.17(.18)	.26(.26)
71/04/03	33	.05	.13	.19(.20)	.28(.28)
71/03/23	33	.06	.14	(.19)	(.28)
70/08/13	15	.11	.20		
70/06/05	20	.08	.18	.18(.18)	.27(.27)
70/03/28	15	.11	.20		
69/02/11	33	.08	.16	.19(.20)	.29(.29)
67/08/15	33	.08	.18	.19(.20)	.28(.28)
67/02/11	5	.08	.18		
63/04/19	33	.08	.16	.18(.18)	.26(.26)
71/01/10	33	.05	.14		
70/07/26	35	.05	.16		
69/11/07	35	.04	.14		
69/02/28	22	.02	.09		
65/06/27	27	.09	.20		
65/02/02	12	.06	.14		
64/03/28	21	.08	.20		

$T/Q_{eff}(P) = 1.0$

$T/Q_{eff}(s) = 4.0$ for values not in parentheses

$T/Q_{eff}(s) = 4.5$ for values in parentheses

TABLE 3.3 EVENT LIST

DEEP EARTHQUAKES

Date	Depth	$\tau_{2\ p}(\omega^{-2})$	$\tau_{2\ p}(\omega^{-3})$	$\tau_{2\ s}(\omega^{-2})$	$\tau_{2\ s}(\omega^{-3})$
63/11/09	600	.08(.10)	$\infty(\infty)$	.11(.15)	.19(.22)
72/02/30	532	.04(.06)	.21(.22)	.07(.12)	.19(.20)
74/03/23	535	.08(.10)	$\infty(\infty)$	.12(.15)	.21(.24)
68/11/04	585	.08(.10)	$\infty(\infty)$	.12(.16)	.20(.22)
75/02/22	375	.08(.08)	$\infty(\infty)$	.12(.15)	.18(.21)

$T/Q(P)$ = .06 for values without parentheses

$T/Q(P)$ = 0.785 for values with parentheses

$T/Q(s)$ = 2.4 for values without parentheses

$T/Q(s)$ = 3.5 for values with parentheses

TABLE 3.4

STATISTICS OF OBSERVED RELAXATION PARAMETERS

(t^* from Model SL1)

	Average τ_2			
	$\tau_{2\ p}(\omega^{-2})$	$\tau_{2\ p}(\omega^{-3})$	$\tau_{2\ s}(\omega^{-2})$	$\tau_{2\ s}(\omega^{-3})$
Shallow Events	.073	.162	.185	.266
Deep and Intermediate Events	.094	.22*	.146	.218

*From event of 72/03/30 only

is to be obtained from the τ_2 determined from this study, the depth dependence of the base t^* function must be known.

Added emphasis to this last conclusion is given by Figure 3.4, where τ_2 is plotted as a function of the base t^* value. The purpose of this test is multifold: (1) Examination of the values of t^* and τ_2 in Tables 3.2 and 3.3 shows a montonic trend in t^* vs τ_2 from deep P waves to shallow S waves, and it is necessary to show that the results presented here are not artifacts of the processing. (2) The effect of base Q model on τ_2 must be investigated. (3) The deep event of 72/03/30 was chosen for the study because it is the only deep event for which ω^{-3} slopes may be obtained in the P-wave spectrum. Since this event yields the only data in the ω^{-3} column, it should be carefully checked. Figure 3.4 shows that t^* vs τ_2 is indeed monotonic increasing, but the curves are distinctly different for P and S waves. The change in τ_2 as a function of wave type is not an artifact. There is, however, an interdependence between t^* and τ_2 such that the change in τ_2 as a function of source depth may be strictly a matter of the different t^* required.

Table 3.4 shows that $\tau_2(P)$ increases with source depth, while $\tau_2(S)$ decreases. The decrease in $\tau_2(S)$ is plotted in Figure 3.4, and closely resembles the change in τ_2 with t^* already described. That is, if the spectra of deep and shallow events have the same shape and see the same frequency dependent Q along the raypath, then the change in $\tau_2(S)$ is simply that required to account for the change in t^*. Tentatively, this result is interpreted as support for the basic similarity between deep and shallow shear body-wave spectra. A reasonable corollary is that P-wave spectra are also similar.

Under the assumption of similarity, the increase in $\tau_2(P)$ with source depth must be interpreted as a mixed depth and frequency dependence in Q_α.
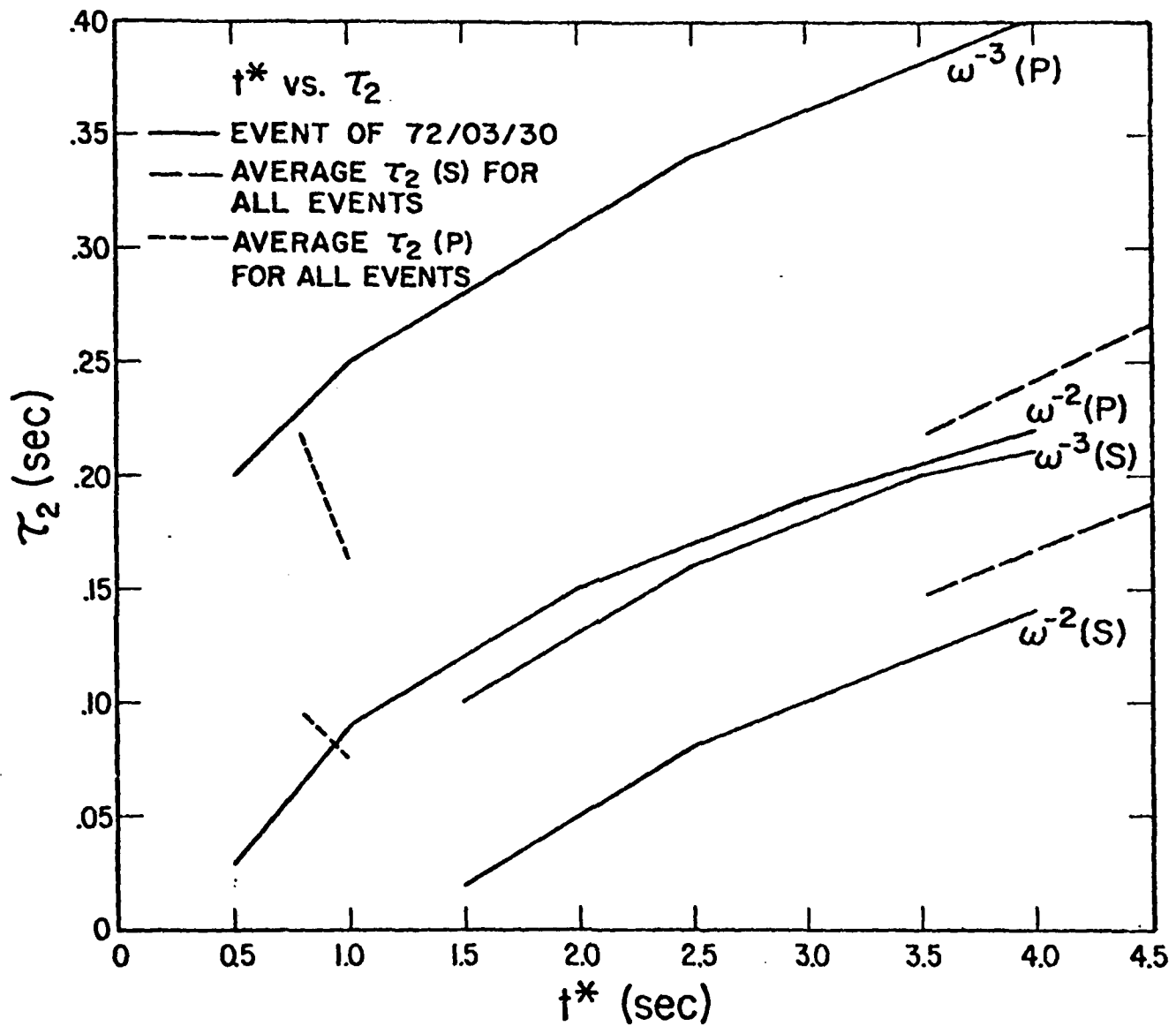
Figure 3.4: τ_2 vs. t^*. The solid lines give the functional dependence of τ_2 on t^*. For the range of t^*, a τ_2 may always be found which will give the desired slope on body wave spectra. The average S wave data show the same trend as a function of changing τ_2, while the average P data oppose the trend, suggesting the two wave types don't see the same depth dependence in Q.

Specifically, an isolated bulk loss mechanism must operate in the upper mantle. The fact that $\tau_2(P)$ increases toward $\tau_2(S)$ implies that the amount of bulk loss seen along the travel path is decreased. An intuitively appealing explanation is that the bulk loss mechanism operates within the asthenosphere where a partial melt is hypothesized.

These conclusions may be visualized in Figure 3.5. The solid line represents the right side of the shear absorption band as it affects P-waves. The dotted line represents the observed P wave absorption band which is a composite of bulk and shear losses. The dashed line represents the hypothetical bulk loss mechanism which controls $\tau_2(P)$ for shallow focus events. For depth of focus beneath the bulk loss region, only half as much bulk loss is seen by a ray, effectively pushing $\tau_2(P)$ toward the limit of the shear absorption band. Note that the implied relaxation time for the bulk-loss mechanism as sketched is about 0.5 sec.

Frequency Dependent Earth Models.

The results of the preceding section provide a starting place in the construction of a frequency dependent Q. In the examples discussed below, $\tau_2(P)$ will be assumed to be controlled by shear mechanisms in the crust and in the mantle below the asthenosphere. In the asthenosphere, where a partial melt may exist, τ_2 is assumed to be controlled by a bulk loss mechanism. The models AFL and SL1 will provide a framework for model evaluation. As an example of the flexibility offered by manipulation of τ_2, preliminary attempts will be made to generate AFL from SL1 by adjusting the position and strength of a bulk loss mechanism in the asthenosphere.

The difference in average t^* levels between AFL and SL1 is easily modeled by adjusting τ_2 without depth dependence. Table 3.5 gives four
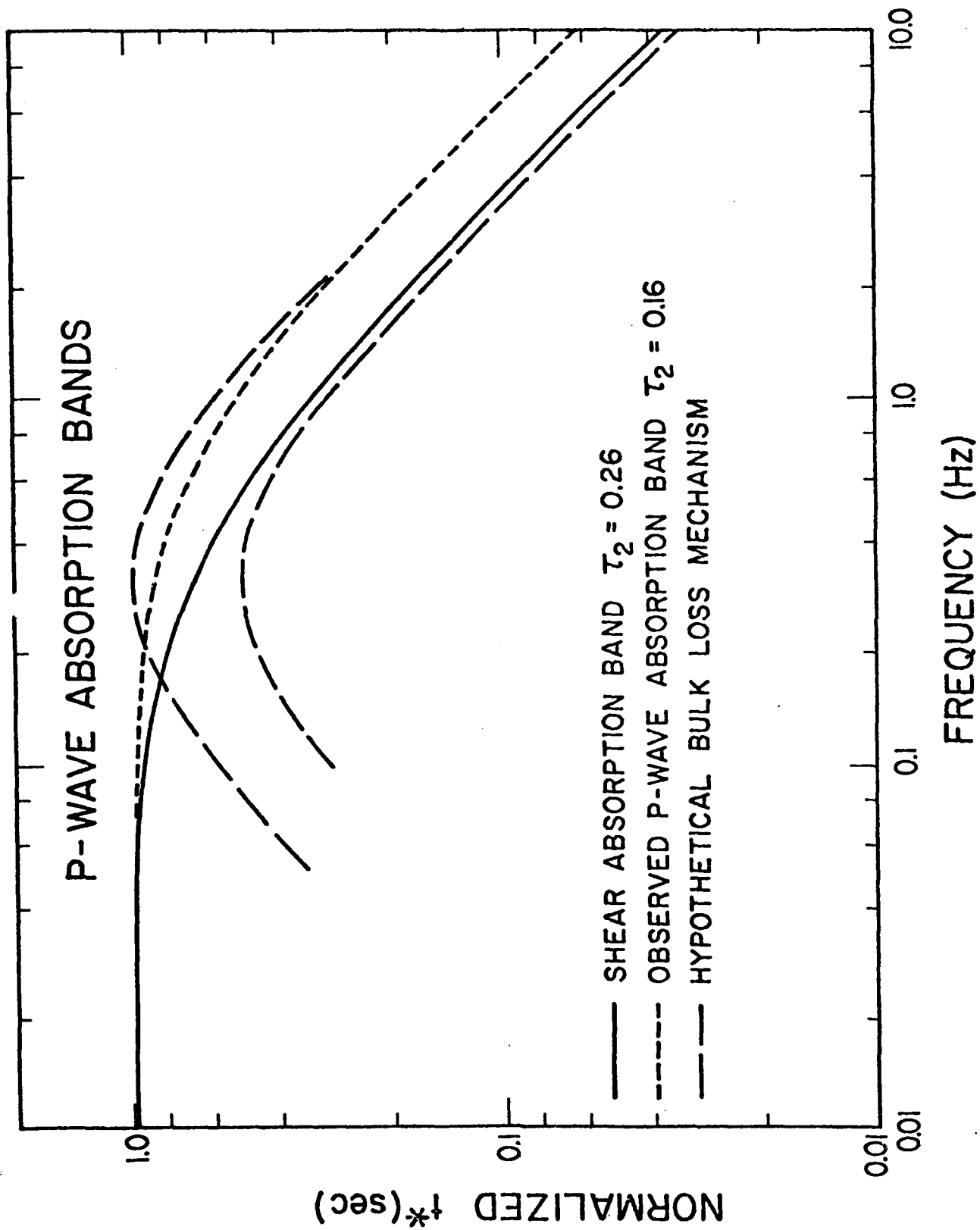
Figure 3.5: t* vs Frequency. The effect of an isolated bulk loss mechanism is shown. For shallow focus events, the bulk loss makes the P-wave absorption band wider than the shear loss absorption band. P waves from deep events see only half as much bulk loss, so that the P-wave and S-wave absorption bands are more nearly the same.

TABLE 3.5

FREQUENCY DEPENDENT Q
MODEL PARAMETERS AND DISTANCE DEPENDENCE

Model 1		Model 2		Model 3		Model 4	
Depth	$\tau_2(P)$	Depth	$\tau_2(P)$	Depth	$\tau_2(P)$	Depth	$\tau_2(P)$
0	0.26	0	0.18	0	0.26	0	0.18
2886	0.26	61	0.18	61	0.26	61	0.12
		81	0.08	81	0.16	81	0.09
		196	0.08	196	0.16	196	0.09
		221	0.18	221	0.26	221	0.10
		2886	0.18	2886	0.26	321	0.12
						371	0.14
						388	0.18
						2886	0.18

	t_α^*		t_α^*		t_α^*		t_α^*	
Δ(DEG)	200 sec	1 sec	200 sec	1 sec	200 sec	1 sec	200 sec	1 sec
80	.800	.284	.771	.400	.770	.274	.772	.415
70	.858	.302	.822	.426	.822	.297	.824	.442
60	.984	.333	.913	.470	.912	.318	.914	.487
50	.942	.327	.918	.472	.917	.350	.919	.490
40	.860	.304	.868	.451	.865	.333	.868	.472
30	.755	.267	.761	.405	.760	.302	.762	.431

Note: t_α^* values are for depth of focus = 5 km.

examples of τ_2 vs depth which have been superimposed on SL1 and the corresponding t^* values for waves of period 200 sec and 1 sec. From Figure 3.5, it is obvious that, for frequencies greater than $\omega = 1/\tau_2$, increasing τ_2 increases the Q . Both $\tau_2(\omega^{-2})$ and $\tau_2(\omega^{-3})$ give approximately correct change in t^* with frequency.

The difference in t^* vs hypocentral depth is not adequately modeled by any of the τ_2 distributions shown in Table 3.5. t^* vs depth is shown in Table 3.6.

The complete inversion for τ_2(depth) required to obtain model AFL from model SL1 will be left to a future report. But it is apparent from the testing already done that the result will be almost trivial. At each depth

$$Q^{-1}(AFL) = Q^{-1}(SL1) \cdot R(\omega, \tau_2)$$

$$\frac{Q(AFL)}{Q(SL1)} = \frac{1}{R(\omega, \tau_2)}$$

Note that the ratio of high frequency Q to low frequency Q must be greater than or equal to one, unless a bulk loss is operating. For these two models, the ratio is less than one for a depth range 0-160 km, and is always greater than two below 160 km. Again, the existence of a bulk loss concentrated in the asthenosphere is supported by the data.

Summary

A review of published Q models was undertaken in an attempt to reduce the number of variables in a study of the effect of frequency dependent Q upon body-wave spectra. However, rather than finding a consensus among Q models, a frequency dependence was found, corresponding to the data set used. For the purposes of further work, t^* will be assumed to follow the

TABLE 3.6

FREQUENCY DEPENDENT Q
SOURCE DEPTH DEPENDENCE
(% Surface Focus t_α^*)

Depth	Model 1		Model 2		Model 3		Model 4	
	200 sec	1 sec	200 sec	1 sec	200 sec	1 sec	200 sec	1 sec
5	100	100	100	100	100	100	100	100
50	97.9	100	99.6	99.6	99.9	100	99.6	99.8
100	94.6	97.2	96.1	95.1	96.1	95.7	95.6	95.3
150	91.0	93.2	92.4	89.8	92.5	90.5	92.3	90.4
200	88.2	90.4	89.8	86.2	89.7	87.1	89.7	86.9
300	83.5	85.7	85.3	82.2	85.2	83.1	85.2	81.4
400	80.7	82.3	81.8	78.8	81.8	80.0	81.7	77.8
600	76.7	78.2	77.2	74.8	77.3	75.7	77.3	73.9

Note: t_α^* values are for $\Delta = 50°$

free-oscillation Q model, SL1. This choice is based upon the low-frequency

data used to compute SL1 and upon the fact that the frequency dependence

used in this study includes no variation in Q at low frequencies.

Given a starting model, then, the approximation of other Q models by

adjustment of τ_2 vs depth is easy. Indeed, any high-frequency Q model

may be generated from any low-frequency model by adjusting the width of the

absorption band. Rather than a single infinity of Q models which will satisfy

a given data set, there is now a double infinity of models, even though the

frequency dependence modeled is simple enough to be characterized by a single

parameter.

It is thus even more important to constrain $Q(f, r)$ with new data.

Of particular importance are the body-wave spectra, where direct observation of

τ_2 may be made. Time-domain modeling of pulse shapes may also improve the

constraint. But basically any ray samples an average of properties along

the path, and detailed inversion for depth dependent properties cannot be

done simply. The conclusion put forth in this paper is that only the end of

a complicated band of relaxation mechanisms may be resolved by body-wave

spectra. If the window in the spectrum of relaxation mechanisms were not in

the passband of standa 1 seismometers, then not even the end of the absorption

band could be observed.

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IV. Applications of Green's Function Methods in Elastodynamics to Source
Theory and Wave Propagation Problems

C.B. Archambeau

A major part of the effort during the past six months of the contract

period has been to develop a comprehensive theoretical framework for the

treatment of both source and wave propagation problems in elastodynamics.

The Appendix 1 provides a summary of the Green's fucntion theory which will

provide such a framework for elastodynamic problems in Geophysics. In

this theoretical development we consider media which may be inhomogeneous,

anisotropic and have both fixed and moving discontinuities in material

properties, the latter being included in order that generalized relaxation

sources in prestressed media may be treated. This work corresponds to a

generalization of previous work (e.g., Archambeau and Minster, 1977)

mentioned in earlier reports, which was specifically focussed on source

problems. The present work is more general in that we have included both

fixed and moving boundaries in the same formulation and most importantly,

have generated explicit Green's functions for layered media to be used in

the integral equations for applications. We are therefore in a position to

solve a large number of outstanding problems in seismic source and wave

propagation theory and specifically particular problems of importance for

seismic event discrimination and explosion yeild estimations.

In this regard, in this section we show first how layered half space

Green's functions are obtained and give explicit relations for them.

We then show how such a Green's function (or more properly this layered

half space Green's tensor) can be used with the integral formulations in the

Appendix 2 to provide the means of representing a general nonlinear (or linear)

energy source imbedded in a layered half space (which may be specified

numerically in general) in analytic form. Such a result can then be used to predict both surface and body wave radiation from general seismic energy sources; in particular from complex numerical models of both earthquakes and explosions.

In addition, we have formulated the same theory in a layered spherical medium. Since this theory is very similar in principal to the layered half space theory we do not include the details here.

Other applications of this theoretical formulation are being studied, in particular effects of lateral variations in structure on predicted source radiation, effects of scattering from the growing source boundary and a variety of perturbation modifications to be used to predict mroe accurately the effects of anelasticity. velocity gradients and boundary variations from spherical or planar form. These additional applications will be described in subsequent reports.

Eigenvectors for Layered, Isotropic Elastic Half-Spaces

Eigenvectors ψ_k corresponding to the operator defined in the Appendix II, equations (61) and (62) are generated by

$$L_{\ell k}\psi_k + \rho\omega^2\psi_\ell = 0 \tag{4.1}$$

$$\left.\begin{array}{l}\left[\!\left[\Psi_{\ell k}n_k\right]\!\right]_{\partial\nu_I} = 0 \\[2ex] \left[\!\left[\psi_\ell\,\right]\!\right]_{\partial\nu_I} = 0 \\[2ex] \left[\Psi_{\ell k}n_k\right]_{\partial\nu_E^{(0)}} = 0\end{array}\right\} \tag{4.2}$$

where $\quad \Psi_{\ell k} \equiv C_{\ell kij}\dfrac{\partial\psi_i}{\partial x_j}$

In a layered space, with boundaries along the coordinate surfaces $x_3 = z_s$, $s = 0,\ldots,n-1$, the set[+]

$$\left\{L_{\ell k}^{(s)}\psi_k^{(s)} + \rho^{(s)}\omega^2\psi^{(s)} = 0\right\}_1^n$$

is equivalent to (63). That is $L_{\ell k}$ is represented by the differential operator set $\left\{L_{\ell k}^{(s)}\right\}_1^n$ and ψ_ℓ by the eigenvector set $\left\{\psi_\ell^{(s)}\right\}$. The members

---

[+]Indices enclosed by parentheses are not subject to the summation rule. Thus no sum is implied on the repeated indices (s) for example.

of the sets are connected by the boundary conditions, which are, from (62):

$$
\left[\begin{pmatrix} \psi^{(s)}_{\ell k} n_k \\ \psi^{(s)}_{\ell} \end{pmatrix}\right]_{z_s} = \left[\begin{pmatrix} \psi^{(s+1)}_{\ell k} n_k \\ \psi^{(s+1)}_{\ell} \end{pmatrix}\right]_{z_s}
$$

$$
\left[\psi^{(1)}_{\ell k} n_k\right]_{z_o} = 0
$$

(4.3)

Here the free surface is the coordinate plane $x_3 = z_o$. These conditions insure that the members of the eigenvector set produce a continuous eigenvector ψ_ℓ with associated continuous tractions. Explicitly, ψ_ℓ is given by

$$
\psi_\ell = \left\{ \psi^{(s)}_\ell \ \Big| \ z_{s-1} \leq z \leq z_s \ \Big| \right\}^n_1
$$

with the adjoining $\psi^{(s)}_\ell$ continuous at the boundary points z_s. A similar statement applies to the tractions.

Now the elastic parameters and density are constant within each layer by definition, so that requiring isotropy as well produces the result:

$$
\left[\lambda^{(s)} + \mu^{(s)}\right] \frac{\partial}{\partial x_\ell} \left(\frac{\partial \psi^{(s)}_k}{\partial x_k}\right) + \mu^{(s)} \frac{\partial^2 \psi^{(s)}_\ell}{\partial x_j \partial x_j} + \rho^{(s)} \omega^2 \psi^{(s)}_\ell = 0
$$

(4.4)

$$
\left[A^{(s)}_{pk} \psi^{(s)}_k\right]_{z_s} = \left[A^{(s+1)}_{pk} \psi^{(s+1)}_k\right]_{z_s} \quad ; \ s = 0,1,\ldots,n-1
$$

where

$$A_{pk}^{(s)} = \begin{pmatrix} B_{\ell k}^{(s)} \\ \\ (1-\delta_{so})\delta_{jk} \end{pmatrix} \; ; \quad p = \begin{pmatrix} \ell \\ j \end{pmatrix} \qquad (4.5)$$

with $B_{\ell k}^{(s)}$ defined by

$$B_{\ell k}^{(s)} \psi_k^{(s)} = \lambda^{(s)} n_\ell \left(\frac{\partial \psi_k^{(s)}}{\partial x_k} \right) + \mu^{(s)} \left(\frac{\partial \psi_j^{(s)}}{\partial x_\ell} + \frac{\partial \psi_\ell^{(s)}}{\partial x_j} \right) n_j \qquad (4.6)$$

which is just the traction $\Psi_{k\ell} n_\ell$. Here the external boundary condition has been incorporated with the boundary conditions on the internal boundaries by defining $\lambda^{(0)} = \mu^{(0)} = \rho^{(0)} = 0$. In addition the 6 x 3 matrix operator $A_{pk}^{(s)}$ has been introduced for convenience in writing the boundary conditions in compact form.

The vector form of

$$\left. \begin{array}{l} \left[\lambda^{(s)} + \mu^{(s)}\right] \nabla\left(\nabla\cdot\underline{\psi}^{(s)}\right) + \mu^{(s)}\nabla \underline{\psi}^{(s)} + \rho^{(s)}\ddot{\underline{\psi}}^{(s)} = 0 \\ \\ \left[\underline{\underline{A}}^{(s)}\cdot\underline{\psi}^{(s)}\right]_{z_s} = \left[\underline{\underline{A}}^{(s+1)}\cdot\underline{\psi}^{(s+1)}\right]_{z_s} \end{array} \right\} \qquad (4.6a)$$

From (4.4) or (4.4a) it is easy to see that solutions for $\underline{\psi}$ may be generated by introduction of the (physical) potentials defined by

$$\left.\begin{array}{l} \chi_4^{(s)} = \dfrac{\partial \psi_\ell^{(s)}}{\partial x_\ell} \\[3em] \chi_j^{(s)} = \dfrac{1}{2}\, \varepsilon_{jk\ell}\, \dfrac{\partial \psi_\ell^{(s)}}{\partial x_k}\;;\quad j = 1,2,3 \end{array}\right\} \tag{4.7}$$

So that these potentials can be represented by a four vector with cartesian components $\chi_{(\alpha)}^{(s)}$, $\alpha = 1,2,3,4$. In vector form:

$$\left.\begin{array}{l} \chi_4^{(s)} = \nabla\cdot\underline{\psi}^{(s)} \\[2.5em] \underline{\chi}^{(s)} = 1/2\; \nabla\times\underline{\psi}^{(s)} \end{array}\right\} \tag{4.7a}$$

Using these in (4.4) or (4.4a) shows that if the cartesian components $\chi_{(\alpha)}^{(s)}$ satisfy the scaler Helmholtz equation:

$$\nabla^2 \chi_{(\alpha)}^{(s)} + \left[k_{(\alpha)}^{(s)} \right]^2 \chi_{(\alpha)}^{(s)} = 0\;;\quad \alpha = 1,2,3,4$$

then $\psi_\ell^{(s)}$ is given by

$$\psi_\ell^{(s)} = -\left[k_p^{(s)}\right]^2 \frac{\partial \chi_4^{(s)}}{\partial x_\ell} + 2\left[k_s^{(s)}\right]^2 \epsilon_{\ell mn} \frac{\partial \chi_n^{(s)}}{\partial x_m} \qquad (4.8)$$

where $k_\alpha^{(s)} = \left(k_s^{(s)}, k_s^{(s)}, k_s^{(s)}, k_p^{(s)}\right)$

with

$$k_p^{(s)} = \omega/v_p^{(s)} \quad ; \quad v_p^{(s)} = \sqrt{\frac{\lambda^{(s)} + 2\mu^{(s)}}{\rho^{(s)}}}$$

$$k_s^{(s)} = \omega/v_s^{(s)} \quad ; \quad v_s^{(s)} = \sqrt{\mu^{(s)}/\rho^{(s)}}$$

Alternately, in vector form, $\underline{\psi}^{(s)}$ is generated from $\chi_4^{(s)}$ and $\underline{\chi} = (\chi_1, \chi_2, \chi_3)$ by

$$\underline{\psi}^{(s)} = -\left[k_p^{(s)}\right]^2 \nabla \chi_4 + 2\left[k_s^{(s)}\right]^2 \nabla x \underline{\chi} \qquad (4.8a)$$

provided the potentials χ_4 and $\underline{\chi}$ satisfy the scalar and vector Helmholtz equations respectively. That is, when

$$\left.\begin{array}{l} \nabla^2 \chi_4^{(s)} + \left[k_p^{(s)}\right]^2 \chi_4^{(s)} = 0 \\[12pt] \nabla^2 \underline{\chi}^{(s)} + \left[k_s^{(s)}\right]^2 \underline{\chi} = 0 \end{array}\right\} \qquad (4.9)$$

Eigenfunctions for the cartesian components of the potentials are the set

$$\left[J_m(k\rho) \ e^{im\phi} \right] e^{-\nu^{(s)}_{(\alpha)} \ z}$$

and

$$\left[J_m(k\rho) \ e^{im\phi} \right] e^{+ \ \nu^{(s)}_{(\alpha)} \ z}$$

where

$$\nu^{(s)}_{(\alpha)} \equiv \sqrt{k^2 - [k^{(s)}_\alpha]^2}$$

It is critical to this development that it turns out that only $\nu^{(s)}_{(\alpha)}$ depends on the layer parameters, as evidenced by the index(s).

Therefore the cartesian components of $\chi^{(s)}_{(\alpha)}$ can be written as:

$$\chi^{(s)}_{(\alpha)} = \left[A^{(s)}_{(\alpha),m}(k,\omega) \ e^{\nu^{(s)}_{(\alpha)} z} + B^{(s)}_{(\alpha),m}(k,\omega) \ e^{-\nu^{(s)}_{(\alpha)} z} \right] J_m(k\rho) \ e^{im\phi} \qquad (4.10)$$

The reason for the introduction of coefficients in the expressions for the χ_α is, of course, that ψ must satisfy boundary conditions in order that it be an eigenvector of the operator defined by (63) and (64), and the coefficients are to be used to satisfy these conditions.

Use of (4.10) in (4.8) produces the cartesian components of $\underline{\psi}^{(s)}$. These can be expressed as components in a cylindrical basis, which is most convenient here. In particular, the eigenvectors can be expressed in terms of the vector cylindrical harmonics $\underline{P}_m$, $\underline{B}_m$ and $\underline{C}_m$ as (Ben Menahem and Singh, 1972; Morse and Feshbach, 1953):

$$\underline{\psi}_m^{(s)}(\underline{r},k) = D_m^{(s)}(z,k)\,\underline{P}_m + E_m^{(s)}(z,k)\,\underline{B}_m + F_m^{(s)}(z,k)\,\underline{C}_m \qquad (4.11)$$

where

$$\left.\begin{aligned}
\underline{P}_m &= \hat{e}_z\, J_m(k\rho)e^{im\phi} \\[2mm]
\underline{B}_m &= \left[\hat{e}_\rho\,\frac{\partial}{\partial(k\rho)} + \hat{e}_\phi\left(\frac{1}{k\rho}\right)\frac{\partial}{\partial\phi}\right]J_m(k\rho)e^{im\phi} \\[2mm]
\underline{C}_m &= \left[\hat{e}_\rho\left(\frac{1}{k\rho}\right)\frac{\partial}{\partial\phi} - \hat{e}_\phi\,\frac{\partial}{\partial(k\rho)}\right]J_m(k\rho)e^{im\phi}
\end{aligned}\right\} \qquad (4.12)$$

Here $\underline{P}_m\cdot\underline{B}_m = \underline{P}_m\cdot\underline{C}_m = \underline{B}_m\cdot\underline{C}_m = 0$.

The coefficient functions of z are (Ben-Menahem and Singh, 1972)

$$
\begin{aligned}
D_m^{(s)}(z,k) &= \nu_p^{(s)} \left[-a_m^{(s)}(k)\, e^{-\nu_p^{(s)} z} + b_m^{(s)}(k)\, e^{\nu_p^{(s)} z} \right] \\[2ex]
&\quad + k \left[c_m^{(s)}(k)\, e^{-\nu_s^{(s)} z} + d_m^{(s)}(k)\, e^{\nu_s^{(s)} z} \right] \\[3ex]
E_m^{(s)}(z,k) &= k \left[a_m^{(s)}(k)\, e^{-\nu_p^{(s)} z} + b_m^{(s)}(k)\, e^{\nu_p^{(s)} z} \right] \\[2ex]
&\quad + \nu_s^{(s)} \left[-c_m^{(s)}(k)\, e^{-\nu_s^{(s)} z} + d_m^{(s)}(k)\, e^{\nu_s^{(s)} z} \right] \\[3ex]
F_m^{(s)}(z,k) &= \left[e_m^{(s)}(k)\, e^{-\nu_s^{(s)} z} + f_m^{(s)}(k)\, e^{\nu_s^{(s)} z} \right]
\end{aligned}
\tag{4.13}
$$

The coefficients $a_m^{(s)}$, $b_m^{(s)}$, etc. are linear combinations of the coefficients appearing in the expressions for the potentials $\chi_{(\alpha)}^{(s)}$.

Note that:

$$
\nu_p^{(s)} = \begin{cases} \sqrt{k^2 - \left(k_p^{(s)}\right)^2} \; ; \; k > k_p^{(s)} \\[2ex] i\sqrt{\left(k_p^{(s)}\right)^2 - k^2} \; ; \; k < k_p^{(s)} \end{cases}
$$

and similarly for $\nu_s^{(s)}$. Also, the radiation condition requires that:

$$b_m^{(n)} = d_m^{(n)} = f_m^{(n)} = 0.$$

The boundary conditions of (4.3) must be satisfied in order that (4.11) be an eigenvector. Using (4.11) in these conditions shows:

(1) that the vector function $\underline{P}_m$, $\underline{B}_m$, $\underline{C}_m$ can be eliminted from the boundary condition equations, so that only the coefficients $D_m^{(s)}(k,z)$, etc. are involved in the boundary conditions.

(2) The boundary equations at each layer interface can be separated into two independent sets, the first composed of the coefficients of $\underline{P}_m$ and $\underline{B}_m$ in   the second involving only the coefficient of $\underline{C}_m$. These correspond to the P-SV coupled waves and the SH waves respectively. They therefore also represent Rayleigh and Love type surface wave modes when the eigenvectors are viewed as modes of oscillation of the medium.

The eigenvector can therefore be viewed as the sum of two decoupled eigenvectors which may be denoted by $^R\underline{\psi}_m^{(s)}$ and $^L\underline{\psi}_m^{(s)}$. Thus

$$\underline{\psi}_m^{(s)} = {}^R\underline{\psi}_m^{(s)} + {}^L\underline{\psi}_m^{(s)}$$

$$\left.\begin{array}{l} {}^R\underline{\psi}_m^{(s)} = D_m^{(s)}\underline{P}_m + E_m^{(s)}\underline{B}_m \\[3em] {}^L\underline{\psi}_m^{(s)} = F_m^{(s)}\,\underline{C}_m \end{array}\right\} \tag{4.14}$$

The traction vectors associated with the eigenvector components $R_{\underline{\psi}_m}(s)$ and $L_{\psi_m}(s)$ on planes parallel to the layer interface boundaries, at $z = z_s$, will be denoted as $R_{\underline{\psi}_m}(s)$ and $L_{\underline{\psi}_m}(s)$ respectively. They are (Ben-Menahem and Singh, 1972):

$$R_{\underline{\psi}_m}^{(s)}(\underline{r},k) = \mathcal{D}_m^{(s)}(z,k)\,\underline{P}_m + E_m^{(s)}(z,k)\,\underline{B}_m$$

$$L_{\underline{\psi}}^{(s)}(\underline{r},k) = F_m^{(s)}(z,k)\,\underline{C}_m$$

$$(4.15)$$

where

$$\mathcal{D}_m^{(s)}(z,k) = 2\mu^{(s)}\left[\left(k^2 - 1/2\left[k_s^{(s)}\right]^2\right)\left\{a_m^{(s)}(k)\,e^{-\nu_p^{(s)}z}\right.\right.$$

$$\left. + b_m^{(s)}(k)\,e^{\nu_p^{(s)}z}\right\} - k\nu_s^{(s)}\left\{c_m^{(s)}(k)\,e^{-\nu_s^{(s)}z} - d_m^{(s)}(k)\,e^{\nu_s^{(s)}z}\right\}\right]$$

$$E_m^{(s)}(z,k) = 2\mu^{(s)}\left[-k\nu_p\left\{a_m^{(s)}(k)\,e^{-\nu_p^{(s)}z} - b_m^{(s)}(k)\,e^{\nu_p^{(s)}z}\right\}\right. \qquad (4.16)$$

$$\left. + \left(k^2 - 1/2\left[k_s^{(s)}\right]^2\right)\left\{c_m^{(s)}(k)\,e^{-\nu_s^{(s)}z} + d_m^{(s)}(k)\,e^{\nu_s^{(s)}z}\right\}\right]$$

$$F_m^{(s)}(z,k) = -\mu^{(s)}\nu_s^{(s)}\left[e_m^{(s)}(k)\,e^{-\nu_s^{(s)}z} - f_m^{(s)}(k)\,e^{\nu_s^{(s)}z}\right]$$

All the boundary conditions are now expressed by the P-SV and SH boundary

conditions. In matrix form they are:

(1) P-SV boundary conditions:

$$
\begin{pmatrix}
D_m^{(s)}(z) \\
E_m^{(s)}(z) \\
\mathcal{D}_m^{(s)}(z) \\
\mathcal{E}_m^{(s)}(z)
\end{pmatrix}_{z_s}
=
\begin{pmatrix}
D_m^{(s+1)}(z) \\
E_m^{(s+1)}(z) \\
\mathcal{D}_m^{(s+1)}(z) \\
\mathcal{E}_m^{(s+1)}(z)
\end{pmatrix}_{z_s}
\tag{4.17}
$$

(2) SH boundary conditions

$$
\begin{pmatrix}
F_m^{(s)}(z) \\
\mathcal{F}_m^{(s)}(z)
\end{pmatrix}_{z_s}
=
\begin{pmatrix}
F_m^{(s+1)}(z) \\
\mathcal{F}_m^{(s+1)}(z)
\end{pmatrix}_{z_s}
\tag{4.18}
$$

The equations involving the unknown coefficients $a_m^{(s)}$, $b_m^{(s)}$,... etc. in (4.13)

and (4.16) may also be written in matrix form. In particular, at $z = z_s$

we have, for the P-SV coefficients:

$$
\begin{pmatrix}
D_m^{(s)}(z) \\
\cdot \\
\cdot \\
\cdot \\
\mathcal{E}_m^{(s)}(z)
\end{pmatrix}_{z_s}
= K_R^{(s)}(z_s)
\begin{pmatrix}
a_m^{(s)} \\
\cdot \\
\cdot \\
\cdot \\
d_m^{(s)}
\end{pmatrix}
\tag{4.19}
$$

and, for the SH terms, similarly:

$$\begin{pmatrix} F_m^{(s)}(z) \\ \\ \mathcal{F}_m^{(s)}(z) \end{pmatrix}_{z_s} = K_L^{(s)}(z_s) \begin{pmatrix} e_m^{(s)} \\ \\ f_m^{(s)} \end{pmatrix} \tag{4.20}$$

The matrices $K_R^{(s)}$ and $K_L^{(s)}$ are 4 x 4 and 2 x 2 respectively, and are invertable (e.g. Haskell, 1953; Harkrider, 1964). Here of course the coefficents $a_m^{(s)} \ldots f_m^{(s)}$ are not functions of z. Therefore we can evaluate the equations at $z = z_{s-1}$, where this is the upper boundary of the layer (s) and get another expression for the constant coefficients. Thus

$$\begin{pmatrix} D_m^{(s)}(z) \\ \vdots \\ \mathcal{E}_m^{(s)}(z) \end{pmatrix}_{z_{s-1}} = K_R^{(s)}(z_{s-1}) \begin{pmatrix} a_m^{(s)} \\ \vdots \\ d_m^{(s)} \end{pmatrix} \tag{4.21}$$

$$\begin{pmatrix} F_m^{(s)}(z) \\ \\ \mathcal{F}_m^{(s)}(z) \end{pmatrix}_{z_{s-1}} = K_L^{(s)}(z_{s-1}) \begin{pmatrix} e_m^{(s)} \\ \\ f_m^{(s)} \end{pmatrix} \tag{4.22}$$

These may be solved for the coefficients $a_m^{(s)} \ldots f_m^{(s)}$ in terms of the functions of z_{s-1} and one has:

$$
\begin{pmatrix} a_m^{(s)} \\ \vdots \\ d_m^{(s)} \end{pmatrix} = \left[K_R^{(s)}(z_{s-1}) \right]^{-1} \begin{pmatrix} D_m^{(s)}(z_{s-1}) \\ \vdots \\ E_m^{(s)}(z_{s-1}) \end{pmatrix}
\tag{4.23}
$$

$$
\begin{pmatrix} e_m^{(s)} \\ \\ f_m^{(s)} \end{pmatrix} = \left[K_L^{(s)}(z_{s-1}) \right]^{-1} \begin{pmatrix} F_m^{(s)}(z_{s-1}) \\ \\ F_m^{(s)}(z_{s-1}) \end{pmatrix}
\tag{4.24}
$$

Now we can eliminate the coefficients from (4.19) and (4.20), this giving us a result which effectively propagates the solution across the layers, from the z_{s-1} boundary to the z_s boundary. In particular, from

$$
\begin{pmatrix} D_m^{(s)}(z_s) \\ \vdots \\ E_m^{(s)}(z_s) \end{pmatrix} = K_R^{(s)}(z_s) \left[K_R^{(s)}(z_{s-1}) \right]^{-1} \begin{pmatrix} D_m^{(s)}(z_{s-1}) \\ \vdots \\ E_m^{(s)}(z_{s-1}) \end{pmatrix}
\tag{4.25}
$$

and from (4.20) using (4.24):

$$
\begin{pmatrix} F_m^{(s)}(z_s) \\ \\ F_m^{(s)}(z_s) \end{pmatrix} = K_L^{(s)}(z_s) \left[K_L^{(s)}(z_{s-1}) \right]^{-1} \begin{pmatrix} F_m^{(s)}(z_{s-1}) \\ \\ F_m^{(s)}(z_{s-1}) \end{pmatrix} \tag{4.26}
$$

Now, let:

$$
\left. \begin{aligned} R^{(s)}(z_s|z_{s-1}) &\equiv K_R^{(s)}(z_s) \left[K_R^{(s)}(z_{s-1}) \right]^{-1} \\ \\ L^{(s)}(z_s|z_{s-1}) &\equiv K_L^{(s)}(z_s) \left[K_L^{(s)}(z_{s-1}) \right]^{-1} \end{aligned} \right\} \tag{4.27}
$$

where these are the Rayleigh and Love or P-SV and SH propagators. Then

$$
\left. \begin{aligned} \begin{pmatrix} D_m^{(s)} \\ \vdots \\ E_m^{(s)} \end{pmatrix}_s &= R^{(s)} \begin{pmatrix} D_m^{(s)} \\ \vdots \\ E_m^{(s)} \end{pmatrix}_{s-1} \\ \\ \\ \begin{pmatrix} F_m^{(s)} \\ F_m^{(s)} \end{pmatrix}_s &= L^{(s)} \begin{pmatrix} F_m^{(s)} \\ F_m^{(s)} \end{pmatrix}_{s-1} \end{aligned} \right\} \tag{4.28}
$$

where the notation has been modified in an obvious way for brevity.

Now the boundary conditions require

$$\begin{pmatrix} D_m^{(s+1)} \\ \vdots \\ E_m^{(s+1)} \end{pmatrix}_s = \begin{pmatrix} D_m^{(s)} \\ \vdots \\ E_m^{(s)} \end{pmatrix}_s = R^{(s)} \begin{pmatrix} D_m^{(s)} \\ \vdots \\ E_m^{(s)} \end{pmatrix}_{s-1}$$

$$\begin{pmatrix} F_m^{(s+1)} \\ F_m^{(s+1)} \end{pmatrix}_s = \begin{pmatrix} F_m^{(s)} \\ F_m^{(s)} \end{pmatrix}_s = L^{(s)} \begin{pmatrix} F_m^{(s)} \\ F_m^{(s)} \end{pmatrix}_{s-1}$$

$$\left.\right\} \quad (4.29)$$

the final equality because of (4.28). Using (4.28) again, with (s) taken as (s-1) in the expression, gives

$$\begin{pmatrix} D_m^{(s-1)} \\ \vdots \\ E_m^{(s-1)} \end{pmatrix}_{s-1} = R^{(s-1)} \begin{pmatrix} D_m^{(s-1)} \\ \vdots \\ E_m^{(s-1)} \end{pmatrix}_{s-2}$$

for example, with a similar expression for the SH propagator equation. This can be used in (4.29) to eliminate the factor

$$\begin{pmatrix} D_m^{(s)} \\ \vdots \\ E_m^{(s)} \end{pmatrix}_{s-1} ,$$

and likewise for the SH equation. Clearly this can be continued until we reach the z_o boundary at the free surface. We get in fact:

$$\begin{pmatrix} D_m^{(s+1)} \\ \vdots \\ E_m^{(s+1)} \end{pmatrix}_s = \left[R^{(s)} R^{(s-1)} \ldots R^{(1)} \right] \begin{pmatrix} D_m^{(1)} \\ \vdots \\ E_m^{(1)} \end{pmatrix}_o$$

$$\begin{pmatrix} F_m^{(s+1)} \\ F_m^{(s+1)} \end{pmatrix}_s = \left[L^{(s)} L^{(s-1)} \ldots L^{(1)} \right] \begin{pmatrix} F_m^{(1)} \\ F_m^{(1)} \end{pmatrix}_o$$

(4.30)

The final boundary condition to be met is the vanishing of tractions at the free surface $z = z_o = 0$, so $D_m^{(1)}(0) = E_m^{(1)}(0) = F_m^{(1)}(0) = 0$ in (4.4.30). On the other hand the displacements are non-zero at $z = 0$. One of them in the P-SV eigenvector may be set to unity for the eigenvector solution sought here, with the eigenvector normalization factors used to account for this later. For the decoupled SH vector, there is only one displacement amplitude and so it may also be set to one, with normalization accounting for it later. Thus we have

$$\begin{pmatrix} D_m^{(s+1)} \\ \vdots \\ E_m^{(s+1)} \end{pmatrix}_s = \left[R^{(s)} \ldots R^{(1)} \right] \begin{pmatrix} 1 \\ \varepsilon_o \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} F_m^{(s+1)} \\ F_m^{(s+1)} \end{pmatrix}_s = \left[L^{(s)} \ldots L^{(1)} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(4.31)

with $\epsilon_o = E_m^{(1)}(0)/D_m^{(1)}(0)$. (Use of ϵ_o as defined amounts to dividing both sides of (4.30)   by $D_m^{(1)}(0)$ for P-SV and by $F_m^{(1)}(0)$ for SH, and then redefining the coefficients so that $D_m'^{(s+1)} = D_m^{(s+1)}/D_m^{(1)}(0)$. Because of our freedom to normalize the eigenvectors however, we can simply write these new coefficients as before.)

Therefore, if this relation is satisfied for s = n-1, then all the boundary conditions are satisfied. (Note that $[R^{(s)}..R^{(1)}]$ must be taken as unity when s = 0.)

The coefficients required to insure that (4.31) is satisfied for all s = 0,...,n-1 are obtained from (4.23-4.24) using (4.31). That is, we require that (4.32) be valid for any s, in particular for s = p-1. The same holds for (4.23) and (4.24) for s = p. Therefore taking s = p - 1 in (4.31) and substituting the resulting relation in (4.23) and (4.24) with s = p used in these, we have

$$
\left.
\begin{aligned}
\begin{pmatrix} a_m^{(p)} \\ \vdots \\ d_m^{(p)} \end{pmatrix}
&= \left[K_R^{(p)}(z_{p-1}) \right]^{-1} \left[R^{(p-1)} R^{(p-2)} \ldots R^{(1)} \right] \begin{pmatrix} 1 \\ \epsilon_o \\ 0 \\ 0 \end{pmatrix} \\[2em]
\begin{pmatrix} e_m^{(p)} \\ f_m^{(p)} \end{pmatrix}
&= \left[K_L^{(p)}(z_{p-1}) \right]^{-1} \left[L^{(p-1)} L^{(p-2)} \ldots L^{(1)} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{aligned}
\right\} \quad (4.32)
$$

when $p \geq 2$.

Defining:

$$J_R^{(p)} = \left[K_R^{(p)}(z_{p-1}) \right]^{-1} \left[R^{(p-1)} \ldots R^{(1)} \right] \; ; \quad n \geq p \geq 2$$

$$J_L^{(p)} = \left[K_L^{(p)}(z_{p-1}) \right]^{-1} \left[L^{(p-1)} \ldots L^{(1)} \right] \; ; \quad n \geq p \geq 2$$

$$J_R^{(p)} = \left[K^{(p)}(z_{p-1}) \right]^{-1} \quad , \quad \text{when } p = 1$$

$$J_L^{(p)} = \left[K_L^{(p)}(z_{p-1}) \right]^{-1} \quad , \quad \text{when } p = 1$$

$$(4.33)$$

then we have:

$$\begin{pmatrix} a_m^{(p)} \\ \vdots \\ d_m^{(p)} \end{pmatrix} = J_R^{(p)} \begin{pmatrix} 1 \\ \varepsilon_o \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} e_m^{(p)} \\ f_m^{(p)} \end{pmatrix} = J_L^{(p)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(4.34)$$

Here J_R^p is a 4 x 4 matrix propagator and $J_L^{(p)}$ a 2 x 2 matrix propagator.
That is:

$$J_R^{(p)} \equiv J_R^{(p)}(z_p | z_o) \quad , \quad J_L^{(p)} \equiv J_L^{(p)}(z_p | z_o)$$

propagate the solution vector from the boundary surface at z_o to the boundary at z_p.

Explicitly, the coefficients are:

$$
\begin{pmatrix} a_m^{(p)} \\[2ex] b_m^{(p)} \\[2ex] c_m^{(p)} \\[2ex] d_m^{(p)} \end{pmatrix} = \begin{pmatrix} \left[J_R^{(p)}\right]_{11} + \epsilon_o \left[J_R^{(p)}\right]_{12} \\[2ex] \left[J_R^{(p)}\right]_{21} + \epsilon_o \left[J_R^{(p)}\right]_{22} \\[2ex] \left[J_R^{p}\right]_{31} + \epsilon_o \left[J_R^{(p)}\right]_{32} \\[2ex] \left[J_R^{p}\right]_{41} + \epsilon_o \left[J_R^{(p)}\right]_{42} \end{pmatrix}
\tag{4.35}
$$

$$
\begin{pmatrix} e_m^{(p)} \\[2ex] f_m^{(p)} \end{pmatrix} = \begin{pmatrix} \left[J_L^{(p)}\right]_{11} \\[2ex] \left[J_L^{(p)}\right]_{21} \end{pmatrix}
\tag{4.36}
$$

With the coefficients satisfying these relations, then the solutions in (4.13) are in fact eigenvectors of the operator of (4.1) and (4.2).

The Layered Half-Space Green's Tensor: Eigenvector Expansion in the Frequency Domain.

The eigenvectors for the layered half space were shown to be given by

$$\underline{\psi} = \left\{ \left. \underline{\psi}_m^{(s)} \; \right| \; z_{s-1} \le z \le z_s \; \left| \; \right\}_1^n \right. \tag{4.37}$$

where (equation 4.13)

$$\psi_m^{(s)} = D_m^{(s)}(z,k) \; \underline{P}_m + E_m^{(s)}(z,k) \; \underline{B}_m + F_m^{(s)}(z,k) \; \underline{C}_m$$

$$\equiv \underline{P}_m^{(s)} + \underline{B}_m^{(s)} + \underline{C}_m^{(s)} \tag{4.38}$$

The functions $D_m^{(s)}$, etc., satisfy the boundary conditions of (4.2) if the set of constant coefficients (i.e., independent of z) appearing in these functions satisfy the conditions of (4.35) and (4.36).

Therefore we have found eigenvectors such that

$$\left. \begin{array}{l} \rho^{-1} L_{\ell k} \psi_k + \omega^2 \psi_\ell = 0 \\[2ex] \left[\!\left[B_{\ell k} \psi_k \right]\!\right]_{\partial \nu_I} = 0 \\[2ex] \left[\!\left[\psi_k \right]\!\right]_{\partial \nu_I} = 0 \\[2ex] \left[\!\left[B_{\ell k} \psi_k \right]\!\right]_{\partial \nu_E} = 0 \end{array} \right\} \tag{4.39}$$

Here the original differential equation has been divided through by ρ,
the density, in order to provide the same differential operator,
$\rho^{-1} L_{\ell k} \equiv L'_{\ell k}$ that was actually used to generate the eigenvectors of (4.37).

The functional orthogonality of the $\psi(\underline{r},k)$ can now be easily demonstrated
and this in turn can be used to obtain the expansion for $\tilde{H}_{\ell}^{m}$ in terms of
these eigenvectors. (Actually, completeness of the eigenvector set is
also needed for such an expansion to be rigorous. However, we will obtain
the expansion without the proof of completeness, just using the orthogonality
of the ψ. Since the $\tilde{H}_{\ell}^{m}$ so generated satisfy both the differential equation
for a Greens function and the required boundary conditions then by
uniqueness of solutions for such an equation we can claim to have a
sufficiently proper representation in the distributional sense.)

The orthogonality of the eigenvectors ψ, viewed as a set composed of
members with different eigenvalues m and k, is a consequence of the self-
adjointness of the operator defined by (all) the equations of (4.39).
(Here m and k are eigenvalues or "quantum numbers" for eigenvectors of
the operator. Clearly m is a discrete set of integers, while k
is a continuous set. For a bounded medium, like a layered sphere, the
eigenvalues appearing are m and ℓ. They correspond (one to one) to the
set m, k appearing in this treatment of an unbounded medium. For the
bounded spherical medium however both m and ℓ are discrete integer sets.
These differences are, as implied, a consequence of the bounded or unbounded
nature of the space over which the operator is defined, the basic operator
being the same in both cases.)

The self-adjointness of (4.39) is demonstrated by considering two

arbitrary, distinct members of the eigenvector set $\underline{\psi}$ and (say) $\underline{\phi}$, where

we use different symbols for these members for clarity. Since $\underline{\psi}$ and $\underline{\phi}$

are distinct they will have two distinct sets of eigenvalues. In particular,

the eigenvectors satisfy:

$$
\left.\begin{aligned}
L_{\ell k}\psi_k &= -\omega^2\psi_\ell \\
L_{\ell k}\phi_k &= -\nu^2\phi_\ell
\end{aligned}\right\} \tag{4.40}
$$

where $L'_{\ell k} = \rho^{-1}L_{\ell k}$ is used, with the distinction of the prime ignored.

Here both ω^2 and ν^2 are real. The inner product of $\underline{\phi}$ with the first of

these gives (see Archambeau and Minster, 1977 for details):

$$
\left(L_{\ell k}\psi_k, \phi_\ell\right)_\nu = \left(\psi_\ell, L^*_{\ell k}\phi_k\right)_\nu
$$

$$
+ \int_\nu J_{k,k}d^3x \tag{4.41}
$$

where

$$
J_k = \overline{\phi}_\ell\left[C_{\ell kij}\partial_j\psi_i\right] - \psi_\ell\left[C_{\ell kij}\partial_j\overline{\phi}_i\right] \tag{4.42}
$$

is the bilinear form. Here $\overline{\phi}_\ell$ denotes the complex conjugate of ϕ_ℓ.

$L^*_{\ell k}$ is the adjoint differential operator and is found, in Appendix II, (49) to be a

self-adjoint differential expression, in that:

$$
L^*_{\ell k} = L_{\ell k} \tag{4.43}
$$

The inner products are defined, for example, by

$$\left(L_{\ell k} \psi_k, \phi_\ell \right)_\nu \equiv \int_\nu \partial_i \left(C_{\ell i k j} \partial_j \psi_k \right) \overline{\phi}_\ell d^3 x$$

Now integrating the last term using the divergence theorem, taking account of the existence of internal boundaries $\partial\nu_I$ in ν, gives:

$$\int_\nu J_{k,k} d^3 x = \int_{\partial\nu_E^{(0)}} \left[\overline{\phi}_\ell \left(B_{\ell i} \psi_i \right) - \psi_\ell \left(B_{\ell i} \overline{\phi}_i \right) \right] da$$

$$+ \int_{\partial\nu_I} \left[\!\left[\overline{\phi}_\ell \left(B_{\ell i} \psi_i \right) - \psi_\ell \left(B_{\ell i} \overline{\phi}_i \right) \right]\!\right] da$$

where $B_{\ell k}$ is the boundary operator defined in equation (68). Now ψ and ϕ are eigenvectors satisfying the boundary conditions of (102), and (100) plus (101) is their explicit form. Thus since

$$\left[\!\left[B_{\ell i} \psi_i \right]\!\right]_{\partial\nu_I} = \left[\!\left[B_{\ell i} \phi_i \right]\!\right]_{\partial\nu_I} = 0$$

$$\left[\!\left[\psi_i \right]\!\right]_{\partial\nu_I} = \left[\!\left[\phi_i \right]\!\right]_{\partial\nu_I} = 0$$

then the integral over the internal boundaries vanishes. Likewise, since

$$\left[B_{\ell i} \psi_i \right]_{\partial \nu_E^{(0)}} = \left[\mathcal{B}_{\ell i} \phi_i \right]_{\partial \nu_E^{(0)}} = 0$$

then the first integral vanishes as well. We therefore have that

$$\int_\nu J_{k,k} d^3 x = 0$$

We have as a consequence that:

$$\left(L_{\ell k} \psi_k, \phi_\ell \right)_\nu = \left(\psi_\ell, L_{\ell k} \phi_k \right)_\nu$$

Since $\underline{\psi}$ and $\underline{\phi}$ satisfy (103), then this gives

$$\left(\omega^2 - \overline{\nu}^2 \right) \left(\psi_\ell, \phi_\ell \right)_\nu = 0$$

But $\omega^2 \neq \overline{\nu}^2$, since $\underline{\psi}$ and $\underline{\phi}$ are distinct, so that we must have[+]:

[+]On occasion we will use $<\psi|\phi>$ to denote inner products as well as (ψ, ϕ).

$$\langle \psi_\ell | \phi_\ell \rangle_\nu = \int_\nu \psi_\ell \bar{\phi}_\ell \, d^3x = 0 \qquad (4.44)$$

Hence $\underline{\psi}$ and $\underline{\phi}$ are orthogonal in both the vector inner product sense and the functional sense. (i.e., the repeated index ℓ requires summation over its range so that we also have a vector inner product on ν.)

The vectors $\underline{\psi}$ of the set are therefore mutually orthogonal, however we do not assume that they are normalized. The normalization factors $N_m^P(k,\omega)$, $N_m^B(k,\omega)$, and $N_m^C(k,\omega)$ may be functions of the eigenvalues m, k and are associated with the vectors $\underline{P}_m$, $\underline{B}_m$ and $\underline{C}_m$ of defined from the magnitude of the inner product:

$$\langle \psi_\ell | \psi_\ell \rangle_\nu \equiv \int_\nu \psi_\ell \bar{\psi}_\ell d^3x$$

In view of the vector orthogonality of $\underline{P}_m$, $\underline{B}_m$ and $\underline{C}_m$, we can define norms for each of the component vectors making up ψ, in particular:

$$\left. \begin{aligned} N_m^P(k) &= \int_\nu \underline{P}_m \cdot \overline{\underline{P}}_m d^3x \\[2mm] N_m^B(k) &= \int_\nu \underline{B}_m \cdot \overline{\underline{B}}_m d^3x \\[2mm] N_m^C(k) &= \int_\nu C_m \cdot \overline{C}_m d^3x \end{aligned} \right\} \qquad (4.45)$$

In view of the fact that $\underline{\psi}$ is the sum of P_m, B_m and C_m defined in (4.37), then the orthogonality of $\underline{\psi}$ is expressed by (using (4.44) and (4.45) together):

$$
\left.
\begin{aligned}
\int_\nu \underline{P}_m \cdot \overline{\underline{P}}'_{m'} \, d^3x &= N^P_m \delta^{m'}_m \delta(k-k') \\[2em]
\int_\nu \underline{B}_m \cdot \overline{\underline{B}}'_{m'} \, d^3x &= N^B_m \delta^{m'}_m \delta(k-k') \\[2em]
\int_\nu \underline{C}_m \cdot \overline{\underline{C}}'_{m'} \, d^3x &= N^C_m \delta^{m'}_m \delta(k-k')
\end{aligned}
\right\}
\qquad (4.46)
$$

The orthogonality of the eigenvectors can be used to provide an explicit expression for the Greens tensor $\tilde{H}^q_p$. In particular we have that $\tilde{H}^q_p$ satisfies an equation of the type

$$
L_{rp}\tilde{H}^q_p + \nu^2 \tilde{H}^q_r = -4\pi \delta^q_r \delta(\underline{r}-\underline{r}_o)
$$

Here ν is the Fourier time transform parameter, corresponding to the angular frequency. Also $\tilde{H}^q_p$ satisfies the boundary condition of the form

We take $\tilde{H}^q_p(\underline{r},\underline{r}_o)$ to be expressible by an expansion in the eigenvector set $\underline{\psi}(\underline{r})$. For brevity we represent the eigenvalue pair m and k by the single symbol λ. Further we write the sum over the set of eigenvectors as a "sum" over the sets of m and k, represented by $\int_\lambda$. Since m is integer and discrete while k is a continuous set then $\int_\lambda$ is equivalent, in this case, to an ordinary sum over the discrete set of m values after integration over the continuous k set. Thus we take:

$$\tilde{H}^q_p(\underline{r};\underline{r}_o) = \int_\lambda A_q(\underline{r}_o;\lambda)\ \psi_p(\underline{r};\lambda) \tag{4.47}$$

where the $A(\underline{r}_o;\lambda)$ are unknown coefficients for the expansion. They may, of course, be functions of $\underline{r}_o$ and λ as indicated.

Substituting this expansion in the differential expression for $\tilde{H}^q_p$ gives

$$\int_\lambda A_q(\underline{r}_o;\lambda)\left[L_{rp}\psi_p + \nu^2\psi_r\right] = -\ 4\pi\delta^q_r\ \delta(\underline{r}-\underline{r}_o)$$

Taking the inner product with another member of the eigenvector set $\underline{\phi}$, and observing that

$$L_{rp}\psi_p = -\ \omega^2\psi_r$$

in the above, we have

$$\int_\lambda A_q(\underline{r}_o;\lambda)\left[(\omega^2-\nu^2)< \psi_r\big|\phi_r>\right] = 4\pi< \delta(\underline{r}-\underline{r}_o)\ \big|\ \phi_q(\underline{r};\lambda') > \tag{4.48}$$

Using this and the orthogonality of the eigenvectors, we can determine the coefficients of $A_q(\underline{r}_o;\lambda)$ in the Green's function expansion (4.47).

However, to obtain normalization factors for the A_q in the Green's function expansion which are consistent with the use of the ratio (ellipticity factor) ε_o in reducing the eigenvector coefficients in equation (4.30) to those of (4.31), we must express the eigenvectors as the sum of a P-SV type vector and a SH vector, in the form $\underline{\psi} = \underline{\psi}^R + \underline{\psi}^L$, and

normalize $\underline{\psi}^R$ and $\underline{\psi}^L$ separately. This decomposition simply means that the equations of motion $L_{rp}\psi_p + \omega^2\psi_r = 0$ and boundary conditions can simultaneously be broken into two independent sets with one set satisfied by $\underline{\psi}^R$, the other by $\underline{\psi}^L$. The analysis starting with equation (4.37) and resulting in equation (4.38) applies to each operator set (differential equation plus boundary conditions) involving $\underline{\psi}^R$ and $\underline{\psi}^L$. Then $\tilde{H}_\ell^m$ is given by

$$\tilde{H}_\ell^m = {}_R\tilde{H}_\ell^m + {}_L\tilde{H}_\ell^m$$

with

$$_R L_{\ell k} \; {}_R\tilde{H}_k^m + \omega^2 \, {}_R\tilde{H}_\ell^m = - \, 4\rho\delta_k^m\delta_R(\underline{r} - \underline{r}_o)$$

$$_L L_{\ell k} \; {}_L\tilde{H}_k^m + \omega^2 \, {}_L\tilde{H}_\ell^m = - \, 4\pi\delta_\ell^m\delta_L(\underline{r} - \underline{r}_o)$$

where $\delta(\underline{r} \cdot r_o) = \delta_R(\underline{r}-\underline{r}_o) + \delta_L(\underline{r}-\underline{r}_o)$, in which δ_R and δ_L are decompositions of δ such that $<\delta_R|\underline{\psi}^L> = 0$ and $<\delta_L|\underline{\psi}^R> = 0$.

We are then led to results parallel in all respects to those already obtained for the eivenvectors; and such that

$$\int_\lambda A_q^R(\underline{r}_o;\lambda) \left[(\omega^2 - \nu^2) < \psi_r^R \mid \phi_r^R > \right] = 4\pi < \delta_R(\underline{r} - \underline{r}_o) \mid \phi_q^R > \quad (4.48a)$$

$$\int_\lambda A_q^L(\underline{r}_o;\lambda) \left[(\omega^2 - \nu^2) < \psi_r^L \mid \phi_r^L > \right] = 4\pi < \delta_L(\underline{r} - \underline{r}_o) \mid \phi_q^L > \quad (4.48b)$$

with $\delta_R(\underline{r} - \underline{r}_o)$ and $\delta_L(\underline{r} - \underline{r}_o)$ being regular delta functions of $\underline{r}$ and $\underline{r}_o$, but with the added property that these are projection operators along $\underline{\psi}^R$ and $\underline{\psi}^L$ (e.g., Morse and Feshbach, 1953). Here:

$$\underline{\psi}^R = \underline{P}_m + \underline{B}_m$$

$$\underline{\psi}^L = \underline{C}_m$$

in terms of our previous definitions. Now we have that:

$$< \psi_r^R \mid \phi_r^R > = \int_\nu \underline{\psi}^R \cdot \underline{\overline{\phi}}^R d^3x = \left[N_m^P + N_m^B \right] \delta_m^{m'} \delta(k-k') \qquad (4.49)$$

$$< \psi_r^L \mid \phi_r^L > = N_m^C \delta_m^{m'} \delta(k-k') \qquad (4.50)$$

where we have expressed ϕ_r^R and ϕ_r^L as

$$\left.\begin{aligned}
\underline{\phi} &= \underline{\phi}^R + \underline{\phi}^L \\
\underline{\phi}^R &= \underline{P}_{m'}' + \underline{B}_{m'}' \\
\underline{\phi}^L &= \underline{C}_{m'}'
\end{aligned}\right\}$$

Further[+]

$$\cdot< \delta_R(\underline{r} - \underline{r}_o) \ \Big| \ \phi_q^R(\underline{r};\lambda') > \ = \ \overline{\phi}_q^R(\underline{r}_o;\lambda')$$

$$< \delta_L(\underline{r} - \underline{r}_o) \ \Big| \ \phi_q^L(\underline{r};\lambda') > \ = \ \overline{\phi}_q^L(\underline{r}_o;\lambda')$$

Therefore (111-a) and (111-b) give

$$A_q^L(\underline{r}_o;\lambda') \ = \ \frac{4\pi}{N^R(\lambda')} \ \overline{\phi}_q^R(\underline{r}_o;\lambda')$$

$$A_q^L(\underline{r}_o;\lambda') \ = \ \frac{4}{N^L(\lambda')} \ \overline{\phi}_q^L(\underline{r}_o;\lambda')$$

Here we have set:

$$\left. \begin{array}{l} N^R(\lambda') \equiv (N_m^P{}' + N_m^B{}')(\omega^2 - \nu^2) \\ \\ N^L(\lambda') \equiv (N_m^P{}')(\omega^2 - \nu^2) \end{array} \right\} \qquad\qquad (4.51)$$

_____

[+] The notation $\overline{\phi}$ indicates the complex conjugate of ϕ.

Since $\underline{\phi} = \underline{\phi}^R + \underline{\phi}^L$ is just one of the eigenvectors of the set and λ' its

associated eigenvalue set, then it is equivalent to write

$$\left. \begin{array}{l} A_q^R(\underline{r}_o;\lambda) = \dfrac{4\pi}{N^R(\lambda)} \ \overline{\psi}_q^R(\underline{r}_o;\lambda) \\ \\ \\ A_q^L(\underline{r}_o;\lambda) = \dfrac{4\pi}{N^L(\lambda)} \ \overline{\psi}_q^L(\underline{r}_o;\lambda) \end{array} \right\} \qquad (4.52)$$

The Green's function is therefore given by

$$\tilde{H}_p^q(\underline{r};\underline{r}_o) = {}_R\tilde{H}_p^q + {}_L\tilde{H}_p^q = 4\pi \int\limits_{\lambda} \left[\frac{\psi_p^R(\underline{r};\lambda) \ \overline{\psi}_q^R(\underline{r}_o;\lambda)}{N^R(\lambda)} \right.$$

$$(4.53)$$

$$\left. + \frac{\psi_p^L(\underline{r};\lambda) \ \overline{\psi}_q^L(\underline{r}_o;\lambda)}{N^L(\lambda)} \right]$$

writing this out more fully, using the appropriate representation of the

generalized sum over λ, we have

$$
\tilde{H}_p^q(\underline{r};\underline{r}_o) = 4\pi \sum_m \int_o^\infty dk \left\{ \frac{1}{N_m^R(k)} \left[\left(D_m(z;k) \quad \underline{P}_m(\rho,\phi;k)\cdot\hat{e}_p \right. \right. \right.
$$

$$
+ E_m(z;k) \; \underline{B}_m(\rho,\phi;k)\cdot\hat{e}_p \bigg) \bigg(\overline{D}_m(z_o;k) \; \overline{\underline{P}}_m(\rho_o,\phi_o;k)\cdot\hat{e}_q
$$

$$
\left. + \overline{E}_m(z_o;k) \; \overline{\underline{B}}_m(\rho_o\phi_o;k)\cdot\hat{e}_q \right) \bigg]
$$

$$
+ \frac{1}{N_m^L(k)} \left[\left(F_m(z;k) \; \underline{C}_m(\rho,\phi;k)\cdot\hat{e}_p \right) \left(\overline{F}_m(z_o;k)\cdot \; \overline{\underline{C}}_m(\rho_o,\phi_o;k)\cdot\hat{e}_q \right) \right] \right\} \qquad (4.54)
$$

Here $\hat{e}_p$ and $\hat{e}_q$ are the p and q components of the coordinate basis chosen.
Of course a cylindrical coordinate basis is the natural choice for this
representation.

The functions D_m, E_m and F_m are defined in equation (4.11) and $\underline{P}_m$, $\underline{B}_m$
and $\underline{C}_m$ in (4.10). The coefficients in the function D_m, etc., are given by
(4.35) and (4.36).

Representation of the Radiation Field from a Linear or Non-linear Energy Source

We have at our disposal the integral representation of the spectrum of a displacement radiation field in the form (equation (60)):

$$4\pi\tilde{u}_q = \int_{\nu\Theta\nu} \rho\tilde{f}_p\tilde{H}_p^q d^3x_o$$

$$-\int_{\partial\nu_E(0)} \left[\tilde{H}_p^q\tilde{\tau}_{pr} - \tilde{u}_p\tilde{H}_{pr}^q\right] n_r da_o \tag{4.55}$$

and the layered half space Green's tensor

$$\tilde{H}_p^q(\underline{r};\underline{r}_o) = 4\pi \sum_m \int_o^\infty dk \left[\frac{\psi_p^R(\underline{r};m,k)\ \overline{\psi}_q^R(\underline{r}_o;m,k)}{N_m^R(k)}\right.$$

$$\left. + \frac{\psi_p^L(\underline{r};m,k)\ \overline{\psi}_q^L(\underline{r}_o;m,k)}{N_m^L(k)}\right] \tag{4.56}$$

Consider now the application of these relations to the representation of the elastic radiation from a general energy source (e.g. non-linear). We will assume that the source is actually given by a detailed numerical calculation of the motion due to some particular phenomena (e.g. an explosion) which is most likely a very non-linear process. Suppose the source region occupies some finite volume within the half space and, for generality, that this source volume or nonlinear zone, may intersect the free surface. The situation may be as shown in Figure 2. In any case ν is the linear elastic zone, which will be assumed to be a layered half space. The external

boundary of ν, denoted as $\partial\nu_E$, can be viewed as being composed of two sections, the non-intersected free surface, $\partial\nu_E^{(0)}$, and the (elastic) boundary between the (non-linear) source region and the elastic zone, $\partial\nu_E^{(1)}$.

With the layer half-space Green's function given in (4.56), and neglecting body forces at least for the moment (i.e. the first integral contribution can always be added later), we have for the elastic displacement field in ν (outside the source region):

$$4\pi\tilde{u}_q = \int_{\partial\nu_E^{(1)}} \left[\tilde{H}_p^q \tilde{\tau}_{pr} - \tilde{u}_p \tilde{H}_{pr}^q \right] n_r da_o \qquad (4.57)$$

We can presume knowledge of both $\tilde{u}_p$ and $\tilde{\tau}_{pr} \cdot n_r$, the tractions, on $\partial\nu_E^{(1)}$, the source boundary, since in the present application we have supposed that the results of a numerical calculation are specified on $\partial\nu_E^{(1)}$ in the form of Fourier transforms of $\tilde{\tau}_{pr} \cdot n_r$ and $\tilde{u}_p$

The question is then, how does this excite the surrounding elastic medium. The answer of course is given by (4.57) in the form of the surface integral which gives $\tilde{u}_p$, the elastic displacement at any point in ν. We note that the integral involves $\partial\nu_E^{(1)}$ only, and that this is only part of the nonlinear zone surface boundary that does not intersect the free surface.

The representation (4.57) can be put in more explicit form. That is using (4.56) in (4.57) and interchanging the integral over the surface $\partial\nu_E^{(1)}$ with the sum over m and integral over k, gives:

$$\tilde{u}_q = -\sum_m \int_0^\infty dk \left\{ \frac{\psi_p^R(\underline{r};m,k)}{N_m^R(k)} \int_{\partial v_E^{(1)}} (\tilde{\tau}_{pr} \, n_r) \, \vec{\psi}_q^R(\underline{r}_o;m,k) \, da_o \right.$$

$$- \frac{\psi_q^R(\underline{r};m,k)}{N_m^R(k)} \int_{\partial v_E^{(1)}} (\tilde{u}_p) \, C_{prst} \frac{\partial}{\partial x_o^t} \, \vec{\psi}_s^R(\underline{r}_o;m,k) \, n_r da_o$$

$$+ \frac{\psi_p^L(\underline{r};m,k)}{N_m^{(L)}(k)} \int_{\partial v_E^{(1)}} (\tilde{\tau}_{pr} \cdot n_r) \, \vec{\psi}_q^L(\underline{r}_o;m,k) \, da_o \qquad (4.58)$$

$$\left. - \frac{\psi_q^L(\underline{r};m,k)}{N_m^L(k)} \int_{\partial v_E^{(1)}} (\tilde{u}_p) \, C_{prst} \frac{\partial}{\partial x_o^t} \, \psi_s^L(\underline{r}_o;m,k) \, n_r da_o \right\}$$

We observe that $\tilde{u}_q$ can, as expected, be written as:

$$\tilde{u}_q = \tilde{u}_q^R + \tilde{u}_q^L \qquad (4.59)$$

Here also it is clear that we can define tractions associated with $\underline{\psi}$ the eigenvectors on the surface $\partial v_E^{(1)}$. (Previously, in equations (4.13) and (4.14), tractions on planes parallel to the layer interfaces were defined and called $\underline{\Psi}_m$. These are the tractions computed for the layer matrix and used in the propagators. They are also therefore those given by the standard numerical calculations, e.g. Harkrider, 1964.) These are undefined until we choose a surface. In this regard we have some flexibility of choice since any boundary shape so long as it's entirely within the linear region around the source zone will do. Clearly a surface with

cylindrical coordinate planes is advantageous. To be definite about the matter we will assume that $\partial v_E^{(1)}$ is taken to be a cylindrical surface. Thus, the eigenvector stresses when denoted as ψ_{pr}^R and ψ_{pr}^L, so that:

$$\left.\begin{aligned} \psi_{pr}^R &= C_{prst} \frac{\partial}{\partial x_o^t} \psi_s^R \\[2em] \psi_{pr}^L &= C_{prst} \frac{\partial}{\partial x_o^t} \psi_s^L \end{aligned}\right\} \tag{4.60}$$

can be used to provide the tractions on $\partial v_E^{(1)}$, as $\psi_{pr}^R \cdot n_r$ and $\psi_{pr}^L \cdot n_r$. These show the explicit dependence on the normal, as well as shorten (4.58). That is, we have

$$\tilde{u}_q^R = -\sum_m \int_0^\infty dk/N_m^R(k) \left[\psi_p^R(\underline{r}) \int_{\partial v_E^{(1)}} (\tilde{\tau}_{pn} \cdot n_r) \, \overline{\psi}_q^R \, da_o \right.$$

$$\left. - \psi_q^R(\underline{r}) \int_{\partial v_E^{(1)}} (\tilde{u}_p) \, \overline{\psi}_{pr}^R n_r \, da_o \right] \tag{4.61}$$

$$\tilde{u}_q^L = -\sum_m \int_0^\infty dk/N_m^L(k) \left[\psi_p^L(\underline{r}) \int_{\partial v_E^{(1)}} (\tilde{\tau}_{pr} \cdot n_r) \, \overline{\psi}_q^L \, da_o \right.$$

$$\left. - \psi_q^L(\underline{r}) \int_{\partial v_E^{(1)}} (\tilde{u}_p) \, \overline{\psi}_{pr}^L \, n_r \, da_o \right] \tag{4.62}$$

where

Therefore $\tilde{u}_q^R$ and $\tilde{u}_q^L$ have the forms:

$$\tilde{u}_q^R = \sum_m \int_0^\infty \left[A_{pq}^R(m,k)\, \psi_p^R(\underline{r};m,k) + B^R(m,k)\, \psi_q^R(\underline{r};m,k) \right] dk \qquad (4.61a)$$

$$\tilde{u}_q^L = \sum_m \int_0^\infty \left[A_{pq}^L(m,k)\, \psi_p^L(\underline{r};m,k) + B^L(m,k)\, \psi_q^L(\underline{r};m,k) \right] dk \qquad (4.62a)$$

where:

$$\left. \begin{array}{l} A_{pq}^{(R,L)} = -\dfrac{1}{N_m^{(R,L)}} \displaystyle\int_{\partial v_E^{(1)}} (\tilde{\tau}_{pr}\, n_r)\, \overline{\psi}^{(R,L)}(\underline{r}_o)\, da_o \\[30pt] B^{(R,L)} = +\dfrac{1}{N_m^{(R,L)}} \displaystyle\int_{\partial v_E^{(1)}} (\tilde{u}_p)\, \overline{\psi}_{pr}^{(R,L)} n_r\, da_o \end{array} \right\} \qquad (4.63)$$

These coefficients are the objects of interest, since once they have been
computed it is a rather routine matter to evaluate (4.61a) and (4.62a)
by standard procedures -- such as by summing residues in the k plane to
obtain surface waves -- and by evaluating the branch line integrals, also
occurring in the k plane, by approximate methods to obtain body waves.
(See Ben-Menahem and Singh, 1972 for examples.)

Since we know the forms of $\psi_q^{(R,L)}$, as given in equations (4.11)-(4.13) plus
(4.35) and (4.36) we can calculate the $\psi_{pr}^{(R,L)} n_r$ components on the surface
in question and then evaluate the integrals in (4.63) by a suitable numerical

integration over the surface, using numerically specified fields $(\tilde{u}_p)$ and $(\tilde{\tau}_{pr} \cdot n_r)$ in the calculation.

Appendix I

Use of the High Order Langer Approximation in Seismogram Synthesis

V.F. Cormier

The objective of this research is to develop a method of seismogram synthesis that is more efficient in computation than existing methods yet capable of predicting waveform modulation by zones of intense vertical gradient of velocity and density. In routine use such a method can facilitate inversion for both source-time functions and earth structure.

The last semi-annual report considered an earth having regions of intense but continuous variations in elastic properties. Representations for teleseismic displacement in this earth were derived using higher order solutions in frequency to the radial eigenfunctions that solve the potential wave equations. Research has been directed now towards the practical evaluation of displacement using this representation. The higher-order solution to the elastic wave equations given recently by Woodhouse (1977) in terms of propagator matrices is shown to be equivalent to the solution derived for the higher order potential equations. It is shown that for practical calculations it is best to describe the earth model as a series of radially inhomogeneous layers, each having a constant raidal gradient in velocity and density. Seismogram synthesis can then be most efficiently achieved using the propagator matrix solution in either a reflectivity or a mode summation method.

<u>Solutions to Potential Wave Equations.</u>

The Fourier-transformed potential wave equations satisfied by P, SV, and SH scalar potentials are given by Richards (1974) as

$$\nabla^2 P + \frac{\rho\omega^2}{\lambda+2\mu}\, P + \epsilon_p P = \frac{K_1}{\rho\omega^2}\left[\frac{\partial^2 P}{\partial r^2} - \frac{\partial}{\partial r}\left(\frac{BV}{r}\right)\right] \qquad (.1a)$$

$$\nabla^2 V + \frac{\rho\omega^2}{\mu}\, V + \epsilon_{SV}\, V = -\frac{K_1}{\rho\omega^2}\left[\frac{\partial P}{\partial r} - \frac{BV}{r}\right] \qquad (.1b)$$

$$\nabla^2 H + \frac{\rho\omega^2}{\mu}\, H + \epsilon_T H = 0 \qquad (.1c)$$

for P, SV, and SH waves respectively, where B is an operator acting as the non-radial part of the Laplacian. Separation of variables allows the solutions to be expressed as a sum over Legendre polynomials:

$$P(r,\omega) = \sum_{n=0}^{\infty} w(r,\, n)\, P_n(\cos\Delta) \qquad (.2a)$$

$$V(r,\omega) = \sum_{n=0}^{\infty} [x(r,n)/(i\omega p)]\, P_n(\cos\Delta) \qquad (.2b)$$

$$H(r,\omega) = \sum_{n=0}^{\infty} y(r,n)\, P_n(\cos\Delta) \qquad (.2c)$$

The constant factor $i\omega p$ ($i = \sqrt{-1}$, ω is radial frequency, p ray parameter) divides the radial eigenfunction for SV waves to recognize its dimensional difference from that of the radial eignefunction for P waves. This difference arises from the definition of P and SV potentials and must be considered when determining the order in frequency of their coupling terms.

Substituting (.2a- .2c) in (.1a- .1c) results in equations for the radial functions:

$$\frac{d^2W}{dr^2} + \omega^2\left[\frac{1}{\alpha^2} - \frac{n(n+1)}{\omega^2 r^2}\right]W + \varepsilon_P W = \frac{K_1}{\rho\omega^2}\left[\frac{d^2}{dr^2}\left(\frac{W}{r}\right) + \frac{n(n+1)}{r^2}\frac{\partial X}{\partial r} + 0\left(\frac{W}{\omega}\right)\right] \quad (\ .3a)$$

$$\frac{d^2X}{dr^2} + \omega^2\left[\frac{1}{\beta^2} - \frac{n(n+1)}{\omega^2 r^2}\right]X + \varepsilon_{SV}X = -\frac{K_1}{\rho\omega^2 r}\left[\frac{r\partial}{\partial r}\left(\frac{W}{r}\right) + n(n+1)\frac{X}{r} + 0\left(\frac{X}{\omega^2}\right)\right] \quad (\ .3b)$$

$$\frac{d^2Y}{dr^2} + \omega^2\left[\frac{1}{\beta^2} - \frac{n(n+1)}{\omega^2 r^2}\right] + \varepsilon_T Y = 0 \quad (\ .3c)$$

where W , X , Y equal rw , rx/(iωp) , ry respectively.

Taking a Liouville transform of eqs. (.3a- .3c), making the variable change proposed by Langer (1949), and substituting ωp-1/2 for n gives the equations:

$$\frac{\partial^2\widetilde{W}}{\partial\xi_P^2} + \frac{\omega^2 Q_P^2}{\theta_P'^2} + \frac{1}{\theta_P'^2}\left[\varepsilon_P + \hat{\delta}_P\right]\widetilde{W} =$$

$$\frac{1}{\theta_P'^2}\frac{K_1 r}{\rho\omega^2}\left[-\frac{\omega^2 Q_P^2}{r}\widetilde{W} + \frac{\omega^2 p^2}{r^2}\frac{d\widetilde{X}}{dr}(\theta_P')^{1/2} + 0(\omega\widetilde{W}) + 0(\omega^2\widetilde{X})\right] \quad (\ .4a)$$

$$\frac{\partial\widetilde{X}}{\partial\xi_S^2} + \frac{\omega^2 Q_S^2}{\theta_S'^2} + \frac{1}{\theta_S'^2}\left|\varepsilon_{SV} + \delta_S\right|X =$$

$$-\frac{K_1}{\theta_S'^2\,\rho\omega^2 r}\left[\frac{\omega^2 p^2 X}{r} + \frac{d\widetilde{W}}{dr}(\theta_S')^{1/2} + 0(\widetilde{X}) + 0(\widetilde{W})\right] \quad (\ .4h)$$

$$\frac{\partial^2 \tilde{Y}}{\partial \xi_S^2} + \frac{\omega^2 \, Q_S^2}{\theta_S'^2} + \frac{1}{\theta_S'^2} \, [\varepsilon_T + \delta_S] \, \tilde{Y} = 0 \qquad (\ .4c)$$

where the prime (') signifies differentiation w.r.t. $\xi_{P,S}$ and

$$Q_P = (\frac{1}{\alpha^2} - \frac{p^2}{r^2})^{1/2} \qquad (\ .4d)$$

$$Q_S = (\frac{1}{\beta^2} - \frac{p^2}{r^2})^{1/2} \qquad (\ .4e)$$

The variables $\xi_{P,S}$ and functions θ_P , θ_S are defined by

$$\xi_P = \theta_P = -\left[\frac{3}{2} \int_{r_P}^{r} Q_P \, dr \right]^{2/3} \qquad (\ .4f)$$

$$\xi_S = \theta_S = -\left[\frac{3}{2} \int_{r_S}^{r} Q_S \, dr \right]^{2/3} \qquad (\ .4g)$$

where r_P and r_S are the respective turning point radii for P and S waves in the medium. The functions ξ_P and ξ_S are the Schwarzian derivatives resulting from the Liouville transformation plus a term reuslting from the substitution for n :

$$\xi = -\frac{\theta'''}{2\theta'} + \frac{3}{4}(\frac{\theta''}{\theta'})^2 + \frac{1}{4r^2}$$

The Schwarzian derivative can be evaluated in terms of the physical variables r , α , β , and p by the identity

$$\delta = -\frac{\theta'''}{2\theta'} + \frac{3}{4}\left(\frac{\theta''}{\theta'}\right)^2 = -\frac{1}{2}\frac{d^2Q}{dr^2} + \frac{3}{4Q^2}\left(\frac{dQ}{dr}\right)^2 + \frac{5Q^2}{36\tau^2}$$

where the subscripts P and S are omitted for evaluation in either the P or S velocity profiles and

$$\tau = \int_{r_{P,S}}^{r} Q_{P,S}\, dr$$

The radial eigenfunctions may now be found to any desired order in frequency using the fundamental series solution given by Olver (1976). For

$$\tilde{W} = Ai(\omega^{2/3}\xi_P)\sum_{m=0}^{\infty}\frac{A_m(\xi_P)}{\omega^{2S}} = \frac{Ai'(\omega^{2/3}\xi_P)}{\omega^{4/3}}\sum_{m=0}^{\infty}\frac{B_m(\xi_P)}{\omega^{2S}} \quad (.5a)$$

$$\tilde{X} = Ai(\omega^{2/3}\xi_S)\sum_{m=0}^{\infty}\frac{\overline{A}_m(\xi_S)}{\omega^{2S}} + Ai'(\omega^{2/3}\xi_S)\sum_{m=0}^{\infty}\frac{\overline{B}_m(\xi_S)}{\omega^{2S}} \quad (.5b)$$

where Ai represents any Airy function solution of the zeroth under equation and A_m, B_m, $\overline{A}_m$, $\overline{B}_m$ are functions to be determined by substitution of the series of eq. .4a and .4b and equating terms of equal order in frequency. Thus it can be determined that A_o is always constant in radius for P, SV, and SH waves and in terms of physical variables B_o

$$\sim \frac{1}{2Q_P}\int_{r_P}^{r}\frac{\epsilon_P + \delta_P + \frac{K_1}{\rho}(Q_P^2 + \sigma_{PS})\, dr}{Q_P} \quad \text{for} \quad P \quad (.6a)$$

$$\equiv \frac{I_P}{2Q_P}$$

$$\sim \frac{1}{2Q_S} \int_{r_P}^{r} \frac{\varepsilon_{SV} + \delta_S + \frac{K_1}{\rho} (\frac{p^2}{r^2} + \sigma_{SP}) \; dr}{Q_S} \qquad \text{for} \quad SV,$$

$$\equiv \frac{I_{SV}}{2Q_S} \qquad\qquad\qquad\qquad (\ .6b)$$

and

$$\sim \frac{1}{2Q_S} \int_{r_S}^{r} \frac{\varepsilon_T + \delta_S \; dr}{Q_S} \qquad \text{for} \quad SH \quad,$$

$$\equiv \frac{I_{SH}}{2Q_{SH}} \qquad\qquad\qquad\qquad (\ .6c)$$

where σ_{PS} , σ_{SP} are coupling coefficients defined by

$$\sigma_{PS} = \frac{p^2}{r} \frac{\partial X_o}{\partial z} (\theta_P' \; \theta_S')^{1/2}/\widetilde{W}_o \qquad\qquad (\ .7a)$$

$$\sigma_{SP} = \frac{1}{2_r} \frac{\partial \widetilde{W}_o}{\partial z} (\theta_S' \; \theta_P')^{1/2}/\widetilde{X}_o \qquad\qquad (\ .7b)$$

in which the subscript (o) denotes the zeroth order term in the solution series for $\widetilde{W}$ and $\widetilde{X}$. After substitution of W_o and X_o above, it can be shown that the coupling coefficients σ_{PS} , σ_{SP} have a strength proportional to p/r . Thus at vertical incidence on a region of intense vertical gradient the coupling coefficients σ_{PS} , $\sigma_{SP} = 0$. This continuum property mimics the non-conversion of P and SV waves vertically

incident on a discontinuity. For non-vertical incidence the coupling
coefficients introduce a phase factor determined by the strength of the
function K_1/ρ between r and the turning point radius. The phase
factor acts to correct for the phase accumulated along the path of a con-
verted wave type from the conversion point. Because the coupling coefficients
require the evaluation of additional Airy functions in the integrand of
integral for the B_0 term, the efficiency of the higher order potential
solution in seismogram synthesis would be poor.

Propagator Solution for Wave Propagation.

An alternative description of seismic wave propagation in an inhomo-
geneous medium can be made in terms of the matrix equation satisfied by
components of displacement and stress (Gilbert and Backus, 1966). The
wave equation is thus written in the form

$$\frac{\partial \underline{\underline{F}}}{\partial r} = s\underline{\underline{M}}(r) \ \underline{\underline{F}}(r)$$

where $\underline{\underline{F}}(r)$ is a funcamental matrix solution whose elements are Fourier-
transformed components of displacement and stress. ($s = -i\omega$ for a forward
Fourier-transform sign of the form $\int e^{+i\omega t} dt$).

In obtaining higher order solutions for F Chapman (1973) decomposed
the matrix M into a sum of matrices of differing order in frequency, i.e.,

$$\underline{\underline{M}} = \sum_{j=0}^{J} \underline{\underline{M}}^{(j)} s^{-j}$$

where $J = 2$ for SH waves and 4 for P or SV waves. Wasow (1965) described
how a uniformly asymptotic solution to such a system can be formulated

similar to the solution eqs.(.5a- .5b) derived by Olver (1976) for a single component system. Using a series of similarity transformations and Langer's (1949) variable change, Chapman (1974) obtained matrix equations analogous to the transformed equations (.4a- .4b),

$$\frac{\partial \widetilde{\underline{\underline{F}}}}{\partial \xi} = \left[\sum_{K=0}^{\infty} \widetilde{\underline{\underline{M}}}^{(K)} (\xi) \ S^{-K} \right] \widetilde{\underline{\underline{F}}} \qquad (.8)$$

and solution formulae analogous to eqs. (.5a- .5b),

$$\widetilde{\underline{\underline{F}}} = \left[\sum_{K=0}^{\infty} \underline{\underline{D}}^{(K)} (\xi) \ S^{-K} \right] \widetilde{\underline{\underline{F}}}_o \qquad (.9)$$

where

$$\widetilde{\underline{\underline{F}}}_o = \begin{pmatrix} Ai(\xi S^{2/3}) & Bi(\xi S^{2/3}) \\ \\ S^{-2/3} \ Ai'(\xi S^{2/3}) & S^{-1/3} \ Bi'(\xi S^{2/3}) \end{pmatrix}$$

(Here the symbol $\sim$ over a matrix denotes that the Langer variable change of eqs. (.4f- .4g) has been made.) Chapman (1974) determined the matrix $\underline{\underline{D}}^{(0)}$ in a flattened earth as

$$\underline{\underline{D}}^{(0)} = \left(\frac{\xi'(Z_a)}{\xi'(Z)} \right)^{1/2} \underline{\underline{I}} \qquad (.10)$$

where $\underline{\underline{I}}$ is the identity matrix and Z_a is a reference depth most conveniently chosen as the earth's surface.

Using Chapman's (1974) results for higher order matrices $\underset{=}{M}^{(K)}$ and Wasow's (1965) theorems, the matrices $\underset{=}{D}^{(1)}$, $\underset{=}{D}^{(2)}$ etc. can be determined. The procedure solves for the functions $q_1^{(i)}$, $q_2^{(i)}$ in solutions of the form

$$\underset{=}{D}^{(0)} = q_1^{(0)} (\xi) \underset{=}{I} + q_2^{(0)} (\xi) \underset{=}{\widetilde{M}}^{(0)} \qquad (.11a)$$

$$\underset{=}{D}^{(1)} = \underset{=}{D}_P^{(1)} + q_1^{(1)} (\xi) \underset{=}{I} + q_2^{(1)} (\xi) \underset{=}{\widetilde{M}}^{(0)} \qquad (.11b)$$

subject to the compatibility conditions

$$\underset{=}{M}^{(0)} \underset{=}{D}^{(K)} - \underset{}{D}^{(K)} \underset{=}{M}^{(0)} = \underset{=}{0} \quad \text{for} \quad K = 0$$

$$= \underset{=}{D}^{(K-1)'} - \sum_{j=1}^{K} \underset{=}{M}^{(j)} \underset{=}{D}^{(K-j)} \text{ for } K \neq 0 \qquad (.12)$$

where $\underset{=}{D}_P^{(1)}$ is a particular solution and (') signifies differention w.r.t. the Langer variable ξ . For example by considering the matrix system for SH waves a particular solution to the compatbility condition for $\underset{=}{D}^{(1)}$ is given in a flattened earth by

$$\underset{=P}{D}^{(1)} = \begin{pmatrix} C & -\dfrac{\dot{\mu}}{2\mu\xi\dot{\xi}} q_1^0 \\ & \\ -\dfrac{\ddot{\xi}}{2\dot{\xi}^2} q_1^0 & C \end{pmatrix} \qquad (.13)$$

where ($\cdot$) signifies differention w.r.t. to depth Z and C is a constant. By again substituting the general solution (11b) for $\underset{=}{D}^{(1)}$ in the compatability condition it can be determined that

$$q_1^{(1)} = -C \qquad\qquad (.14a)$$

and

$$q_2^{(1)} = \frac{1}{\sqrt{\xi\dot\xi}} \int \left[\frac{(\xi m_{11}\, d_{12} + m_{22}\, d_{21})\dot\xi^2 = \dot{d}_{12}\, \xi\dot\xi - \dot{d}_{21}\, \dot\xi - \frac{9q_1^0}{4\xi^2}}{\sqrt{\xi\dot\xi}} \right] dZ \qquad (.14b)$$

where a is the radius of the earth and m_{ij} , d_{ij} denote the

ij elements of the $\underset{=}{M}^0$ and $\underset{=P}{D}^{(1)}$ matrices respectively. Now substituting

this result in eq. (.9) with expressions for the elements of $\underset{=}{M}^{(0)}$,

the solution for the fundamental matric $\underset{=}{\widetilde{F}}$ to second order becomes

$$\underset{=}{\widetilde{F}} = q_1^{(0)}\, \underset{=0}{\widetilde{F}} + \underset{=}{\widetilde{T}}^{(1)}\, \underset{=0}{\widetilde{F}} / s \qquad\qquad (.15)$$

where

$$\underset{=}{\widetilde{T}} = \begin{pmatrix} 0 & d_{12} + q_2^{(1)} \\ d_{21} + \xi\, q_2^{(1)} & 0 \end{pmatrix}$$

When the quantity $d_{12} = q_2^{(1)}$ is expressed in a single integral I_{SH}

over depth, the quantity $d_{21} + \xi\, q_2^{(1)}$ expressed in terms of I_{SH} , and

the varible of integration changed from the flattened depth coordinate

Z to radius r it can be shown that

$$d_{12} + q_2^{(1)} \sim \frac{-1}{2Q_S} \int_{r_S}^{r} \frac{\varepsilon_T + \delta_S}{Q_S} \frac{dr}{} \equiv \frac{-I_{SH}}{2Q_S} \qquad (.16a)$$

and

$$d_{21} + \xi q_2^{(1)} \sim \frac{-1}{\xi^{3/2}} \left[\frac{Q_S \, I_{SH}}{2} + \frac{1}{2} \frac{\dot{\mu}}{\mu} + \frac{\dot{\xi}}{\xi} \right] \qquad (\ .16b)$$

The results given by (.16a- .16b) for a spherical layer are equivalent
to those obtained by Woodhouse (1977) for SH waves in plane layer. Note
that the integral I_{SH} is the same as that derived for the higher order
potential term B_0 in eq. .6c for SH waves.

Practical Evaluation of Seismic Displacement

By Watson transforming the partial wave expansion the seismic dis-
placement components u^P , u^{SV} , u^{SV} can be represented in the frequency
domain as

$$u_r^P (r, \Delta_o, \phi, \omega) = - \frac{i\omega^3 \, M_o}{4\pi\rho_o \alpha^4} \int_\Gamma F^P \, w^{(1)}(r_o) \, w'^{(1)}(r) \, Q^{(2)} \, dp \qquad (\ .17a)$$

$$u_\Delta^{SV} (r_o, \Delta_o, \phi, \omega) = \frac{i\omega^3 \, M_o}{4\pi\rho_o \beta_o^4} \int_\Gamma F^{SV} \, x^{(1)}(r_o) \, x'^{(1)}(r_o) \, Q^{(2)} dp \qquad (\ .17b)$$

$$u_\phi^{SH} (r, \Delta_o, \phi, \omega) = \frac{i\omega^3 \, M_o}{4\pi\rho_o \beta_o^4} \int_\Gamma F^{SH} \, y^{(1)}(r_o) \, y'^{(1)}(r_o) \, Q^{(2)} \, dp \qquad (\ .17c)$$

for a shear dislocation centered at radius r_o with seismic moment M_o
and a radiation pattern described by the factors F^P , F^{SV} , and F^{SH} .
For a continuous earth with regions of intense gradient the last semi-annual
report considered the feasibility of calculating displacement by inte-
gration in the complex p plane using the respresentations given by
eqs. (16a- .16b). If the regions of intense gradient are sufficiently

deep, the Airy functions in the higher order representation for the radial

eigenfunctions w, x, y may be replaced by their asymptotic approximations.

Thus w, x, y are approximated by

$$\begin{matrix} (1) \\ w \\ (2) \end{matrix} \approx \frac{\alpha_o^{1/2} \ e^{\mp i\pi/4} \ e^{\pm \tau} p}{r\omega Q_p} \ \left\{ 1 \pm \frac{i}{\omega} \ [\frac{I_P}{2} - \frac{5}{72\tau_p}] \right\} \qquad (.18a)$$

$$\frac{\begin{matrix} (1) \\ x \\ (2) \end{matrix}}{-i\omega p} \approx \frac{\beta_o^{1/2} \ e^{\mp i\pi/4} \ e^{\pm i\tau} s}{r\omega \ Q_S} \ \left\{ 1 \pm \frac{i}{\omega} \ [\frac{I_{SV}}{2} - \frac{5}{72\tau_s}] \right\} \qquad (.18b)$$

$$\begin{matrix} (1) \\ y \\ (2) \end{matrix} \approx \frac{\beta_o^{1/2} \ e^{\mp i\pi/4} \ e^{\pm i\tau} s}{r\omega \ Q_S} \ \left\{ 1 \pm \frac{i}{\omega} \ [I_{SH} - \frac{5}{72\tau_s}] \right\} \qquad (.18c)$$

for (1) up or (2) downgoing waves.

The representations given by eqs. (.17) and (.18), however, are

additionally limited in practical use to situations in which the coupling

coefficient present in I_P, I_{SH}, and I_{SV} can be neglected. The

coupling coefficients themselves involve the evaluation of Airy functions

of both small and large arguments, making the evaluation of I_P and I_{SV}

very inefficient. A far better representation in practical use for P-SV

displacements is given by the higher order fundamental matrices determined

by Woodhouse (1977). The higher order matrix correction to the fundamental

matrix for P and SV waves fully accounts for wave-type conversion and in-

volves integrals no more complicated than that of the I_{SH} type. A given

displacement component observed at the earth's surface can be calculated

from the product of propagator matrices for each model layer with velocity

and density describable by functions analytic in velocity. For example,

the propagator matrix from the boundary radii r_2 to r_1 of such a layer

can be written as

$$\underline{\underline{P}}(r_2, r_1) = \underline{\underline{F}}(r_2)\, \underline{\underline{F}}^{-1}(r_1) \qquad\qquad (\ .19)$$

$$\sim r_2^{-1}\, \underline{\underline{R}}(r_2)\, (1 + i\omega^{-1}\, r_2\, \underline{\underline{T}}^{(1)}(r_2))\, \underline{\underline{\phi}}(r_2)\, \underline{\underline{G}}(r_2)$$

$$\cdot\ \underline{\underline{G}}^{-1}(r_2)\ \underline{\underline{\phi}}^{-1}(r_1)\ (1 - i\omega^{-1}\, r_1\, \underline{\underline{T}}^{(1)}(r_1))\, \underline{\underline{R}}^{-1}(r_1)\, r_1$$

Woodhouse (1977). The elements of $\underline{R}$ are simple functions of elastic constants density and ray parameter. For practical evaluation, the matrix products $\underline{\underline{\phi}}\ \underline{\underline{G}}$ and $\underline{\underline{G}}^{-1}\ \underline{\underline{\phi}}^{-1}$ can be expressed as matrices containing up and downgoing generalized cosines and radial eigenfunctions using the definitions of Richards (1976)

$$\underline{\underline{\phi}}\ \underline{\underline{G}} = \begin{bmatrix} -\dfrac{\acute{\cos j}}{\beta} & \dfrac{\grave{\cos j}}{\beta} \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} y^{(1)} & 0 \\ 0 & y^{(2)} \end{bmatrix} \qquad (\ .20a)$$

and

$$\underline{\underline{G}}^{-1}\ \underline{\underline{\phi}}^{-1} = \begin{bmatrix} y^{(2)} & 0 \\ 0 & y^{(1)} \end{bmatrix} \cdot \begin{bmatrix} -1 & \dfrac{\acute{\cos j}}{\beta} \\ 1 & \dfrac{\grave{\cos j}}{\beta} \end{bmatrix} \qquad (\ .20b)$$

where

$$\acute{\cos j} = \frac{\beta y^{'(1)}}{+i\omega y^{(2)}}$$

$$\grave{\cos j} = \frac{\beta y^{'(2)}}{-i\omega y^{(2)}}$$

Fuchs and Muller (1971) describe how a displacement component observed at the earth's surface can be calculated using an integral representation equivalent to that eqs. (.17a- .17c) but in which a reflectivity function calculated from a product of propagator matrices substitutes for the radial eigenfunctions w , x or y . An alternative representation of displacement does not Watson transform the partial wave series and instead takes a truncated sum of the discrete modes $n = \omega p - 1/2$ (Sato and Usami, 1963). This representation allows synthesis of a longer portion in time of the seismogram from body to surface waves. The use of Woodhouse's (1977) propagator matrices in either method eliminates the need for describing the earth model as a stack of many plane homogeneous layers. Langer's approximation to radial eigenfunctions, embedded in the Airy functions in the $\underline{\Phi}\ \underline{G}$ and $\underline{G}^{-1}\ \underline{\Phi}^{-1}$ matrices, fully accounts for velocity gradients and boundary curvature of radially inhomogeneous layers. A matrix solution method is capable of handling mode conversion more efficiently than the potential solution.

Since layers may be inhomogeneous, a question then remains of how many inhomogeneous layers are necessary to describe an earth model. In principle the earth model may be completely continuous. At sufficiently high frequencies the next higher order term for the fundamental matrix accounts for the effect of regions in which velocity and density gradients are large and rapidly changing. In practice, however, it is seen that it is best not to represent a region of rapidly changing velocity and density gradient as a single inhomogeneous layer, e.g., representing a low velocity zone by a parabolic function or a transition zone by a hyperbolic tangent function. In these cases the integrand of the integrals I_{SH} etc. in the higher order

matrix would be large and rapidly changing at those depths at which the velocity and density gradients are rapidly changing (Figure 8). One must choose between either increasing the number of inhomogeneous layers in the earth model, and thereby the number of fundamental matrices to be evaluated, or increasing the computation time and magnitude of the higher order correction to the fundamental matrices. Representation of the earth model as inhomogeneous layers of nearly constant depth gradient in velocity and density clearly most efficiently compromises between these choices. This representation minimizes the effect of the higher order correction to the fundamental matrix and in many cases allows the integrals I_{SH} etc. to be evaluated using the three point evaluation scheme described by Jeffreys and Jeffreys (1956).

In the next research period a practical computation code will be developed incorporating higher propagator matrices in a either a reflectivity or mode summation method of seismogram synthesis. The method will be tested in speed and accuracy against other exist , methods of seismogram synthesis.
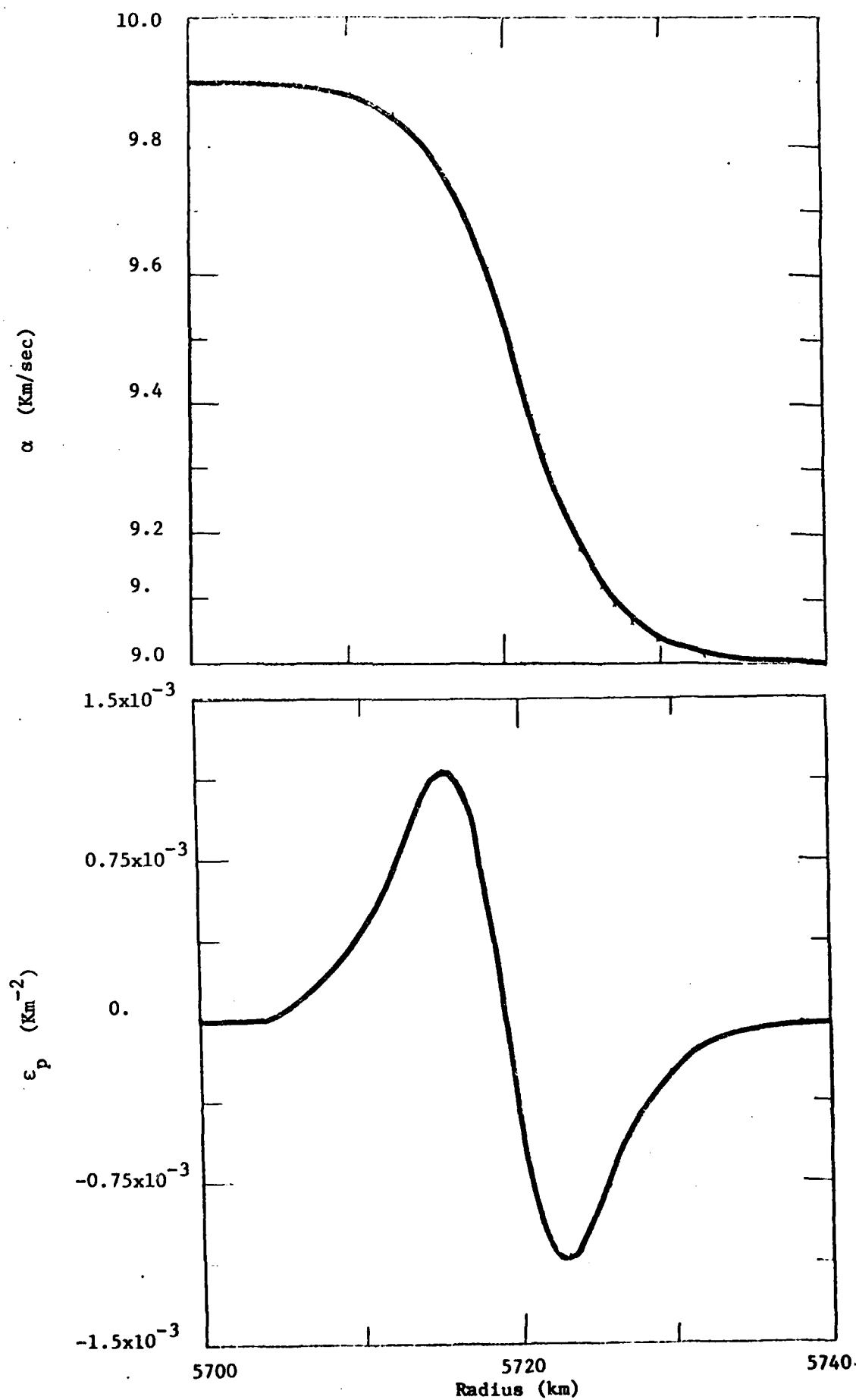
Figure 2..: Behavior of velocity and the higher order H potential term ϵ_T for an Epstein-type transit. zone in velocity.

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Appendix 2: General Elastodynamics Representation Theory for Inhomogeneous
Media with Moving and Fixed Internal Boundaries.

(1) <u>Conservation Equations in a Linear Elastic Medium</u>

The linearized equations of motion may be written as[+] (Archambeau and
Minster, 1977):

$$L_{\alpha\gamma}u_\gamma = \rho f_\alpha \; ; \quad \alpha,\gamma,\delta = (1,2,3,4) \tag{1}$$

Here, u_γ is the space-like displacement field

$$u_\gamma \equiv (u_1,u_2,u_3,0)$$

while f_α is the space-like force density

$$f_\alpha \equiv (f_1,f_2,f_3,0)$$

and $\underline{x}$ is the four vector:

$$x_\gamma \equiv (x_1,x_2,x_3,x_4) \equiv (x_1,x_2,x_3,t)$$

The elastic operator is given by:

$$L_{\alpha\gamma} \equiv \frac{\partial}{\partial x_\beta} \left(C_{\alpha\beta\gamma\delta} \frac{\partial}{\partial x_\delta} \right) \tag{2}$$

[+]The summation convention for repeated indices applies throughout. Indices
excluded from this rule will be enclosed in parentheses, e.g., $G_{ij}^{(n)} u_j^{(n)}$ implies
no sum on n, but summation on the j index over its range of values (1,2,3).
Also, Latin indices will range from 1 to 3, Greek indices from 1 to 4.
Cartesian tensors are used throughout, so while subscripts and superscripts
are used as is notationally convenient, they do not denote covariant or
contravariant components.

with $C_{\alpha\beta\gamma\delta}$ the elastic-inertial tensor:

$$
C_{\alpha\beta\gamma\delta} : \begin{cases} C_{\alpha\beta\gamma\delta} = - C_{ijkl} \quad ; \quad i,j,k,l = (1,2,3) \\[2mm] C_{14k4} = C_{414k} = \rho\delta_{ik}; i,k = (1,2,3) \\[2mm] C_{\alpha\beta\gamma\delta} = 0 \quad ; \quad \text{otherwise} \end{cases} \tag{3}
$$

with C_{ijkl} the usual elastic tensor. For an isotropic elastic medium

$$ C_{ijkl} = \lambda\delta_{ij}\delta_{lk} + \mu(\delta_{il}\delta_{ik} + \delta_{ik}\delta_{il}). $$

C_{ijkl} and $C_{\alpha\beta\gamma\delta}$ obey the same symmetry conditions in all cases, namely:

$$ C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} $$

and

$$ C_{\alpha\beta\gamma\delta} = C_{\beta\alpha\gamma\delta} = C_{\alpha\beta\delta\gamma} = C_{\gamma\delta\alpha\beta} $$

An alternate form of the equations of motion is obtained by defining the generalized inertial-stress tensor:

$$ \tau_{\alpha\beta} \equiv C_{\alpha\beta\gamma\delta} \frac{\partial u_\gamma}{\partial x_\delta} \tag{4} $$

Then, from (1), we have the equation of motion[+]

$$\tau_{\alpha\beta,\beta} = \rho f_{\alpha} \tag{5}$$

Finally, the natural boundary conditions in elastodynamics may be expressed in terms of continuity conditions involving $\tau_{\alpha\beta}$. Thus, if we consider general media in which we admit the possibility of moving boundaries, such as a growing failure zone or moving phase boundaries, etc., then the boundary conditions can be expressed in compact form if we define a "space-time normal" to the surface as a four vector n_{β}, where

$$n_{\beta} \equiv (n_1, n_2, n_3, - U_{\ell}^* n_{\ell}) \tag{6}$$

where $\hat{n} = (n_1, n_2, n_3)$ is the ordinary spatial normal to the boundary surface and U_{ℓ}^* is defined as

$$U_{\ell}^* = U_{\ell} - v_{\ell} \tag{7}$$

where $\underline{v}$ is the particle velocity within the medium into which the normal to the surface is directed and $\underline{U}$ is the velocity of the boundary surface.

The boundaries within and enclosing the medium will be designated by

---

[+]The usual notational convention for partial derivatives will also be used occasionally, that is:

$$\partial_{\beta}\tau_{\alpha\beta} \equiv \tau_{\alpha\beta,\beta} \equiv \frac{\partial \tau_{\alpha\beta}}{\partial x_{\beta}}$$

the symbol Σ. The boundary condition expressing conservation of momentum across any interface or exterior boundary of the medium is (Archambeau and Minster, 1977)[+]

$$[\![\tau_{\alpha\beta} n_\beta]\!]_\Sigma = 0 \tag{8}$$

written out in component form this becomes, using the definitions of $\tau_{\alpha\beta}$, and n_β:

$$[\![(\rho v_k v_\ell^* - T_{k\ell}) \, n_\ell]\!]_\Sigma = 0 \tag{8-a}$$

where

$$v_\ell^* = v_\ell - U_\ell$$

is the particle velocity relative to the moving boundary. The tensor $T_{\ell k}$ is the ordinary Cauchy stress tensor.

Clearly, if the boundary moves with the particles of the medium, in the sense that $U_\ell^* n_\ell = 0$, then $v_\ell^* n_\ell = 0$ also, and the equations (8) reduce to the regular continuity of traction conditions across Σ, that is:

$$[\![T_{k\ell} n_\ell]\!]_\Sigma = 0 \tag{8-b}$$

[+] The double bracket notation $[\![F]\!]_\Sigma$ is used to denote the change in a function F on crossing a surface Σ. That is:

$$[\![F]\!]_\Sigma = F(\Sigma_1) - F(\Sigma_2)$$

The remaining boundary conditions express conservation of mass and energy. They are (Archambeau and Minster, 1977):

$$[\![v_\ell^* n_\ell]\!]_\Sigma = 0 \tag{9}$$

and

$$[\![(\rho E v_i^* - v_k T_{ki} + q_i)\, n_i]\!]_\Sigma = 0 \tag{10}$$

Here q_ℓ is the heat flux and E is the total energy

$$\rho E = \rho u + \rho/2 v_\ell v_\ell + \rho \phi$$

where u is the specific internal energy and ϕ the body force density potential, so, $f_\alpha = - \dfrac{\partial}{\partial x_\alpha} \phi$.

If the boundary moves with the material so that the boundary is carried normal to itself with the particles (i.e., $U_\ell^* n_\ell = 0$), then (9) reduces to continuity of the normal particle velocity across Σ, so:

$$[\![v_\ell n_\ell]\!]_\Sigma = 0 \tag{9-a}$$

In this case (10) reduces to

$$[\![v_k]\!]_\Sigma\, t_k = [\![q_\ell n_\ell]\!]_\Sigma \tag{10-a}$$

where $t_k = T_{k\ell} n_\ell$ are the components of traction. This relation simply expresses the rate of heat production on the boundary (in terms of a jump in heat flux) if there is relative slip of the material on opposite sides

of the boundary. This slip is constrained to be along the boundary, however, because of the constraint imposed by (9-a).

If the physical boundary is such that a no slip condition is warranted - that is $[v_k]_\Sigma = 0$ - then (10-a) reduces to:

$$[q_\ell n_\ell]_\Sigma = 0 \tag{10-b}$$

This just states that the heat flux must be continuous across the boundary. Since thermal effects are of second order in such a situation ("welded" boundaries), then this boundary condition may be neglected in solving for the field u_α from the equations of motion. The boundary condition of importance, in addition to (8-b), is then the condition of no slip, which requires:

$$[v_k]_\Sigma = 0 \tag{11}$$

If this continuity condition is satisfied, then (9-a) is automatically satisfied. Usually (11) is expressed more strongly through the use of the (sufficient) displacement continuity condition. In particular

$$[u_k]_\Sigma = 0 \tag{11-a}$$

It is important to note that (11) and (11-a), amount to assumptions regarding the physical processes at the boundary and are not necessary conditions, the most general case being covered by the relations (8), (9) and (10).

However, it is often the case that the boundary may be considered to be "welded" (for solid-solid interfaces) and then (8-b) and (11) are the

appropriate boundary conditions.

In the cases to be treated here we are concerned with the simpler case of welded internal boundaries (solid-solid contacts). In this case we consider solutions of the system:

$$
\left.\begin{array}{l}
L_{\alpha\gamma}u_{\gamma} = \rho f_{\alpha} \\[4pt]
[\![B_{\alpha\gamma}u_{\gamma}]\!]_{\Sigma} = 0 \\[4pt]
[\![u_{\ell}]\!]_{\Sigma} = 0
\end{array}\right\}
\tag{12}
$$

where the "boundary operator" $B_{\alpha\gamma}$ is defined as:

$$
B_{\alpha\gamma} \equiv n_{\beta}\left[C_{\alpha\beta\gamma\delta}\,\frac{\partial}{\partial x_{\delta}} \right]
\tag{13}
$$

On external boundaries or solid-fluid interfaces the displacement condition is either absent or replaced by:

$$
[\![v_{\alpha} n_{\alpha}]\!]_{\Sigma} = 0
\tag{14}
$$

if one or both materials are fluid.[+] Since the condition that $U^{*}_{\ell}n_{\ell} = - v^{*}_{\ell}n_{\ell} = 0$ applies (the motion of the boundary normal to itself is with the material particles) then the boundary conditions in (12) reduce to the ordinary continuity conditions for traction and displacement.

[+] In the linearized elastodynamic theory, boundary motion is neglected in the sense that all fields are evaluated at the undeformed boundary position. For an external boundary this leads to neglecting the continuity of normal velocity condition.

The boundary conditions expressed in (12) may also be written in the matrix operator form:

$$\left[\!\left[\begin{pmatrix} B_{\alpha\gamma} \\ \\ \delta_{\mu\gamma} \end{pmatrix} u_\gamma \right]\!\right]_\Sigma = 0 \tag{15}$$

(2) Green's Tensors for the Linear Elastic Medium

The Green's tensor associated with the elastic operator $L_{\alpha\gamma}$ can be defined by:

$$L_{\alpha\gamma} G_\gamma^\beta (\underline{x};\underline{x}_o) = \Delta_\alpha^\beta (\underline{x},\underline{x}_o) \tag{16}$$

Here Δ_α^β is the generalized delta function

$$\Delta_\alpha^\beta \equiv 4\pi\, \delta_{\alpha\beta}(1-\delta_{\alpha 4})(1-\delta_{\beta 4})\, \delta(\underline{x}-\underline{x}_o) \tag{16-a}$$

As before the coordinate vector $\underline{x}$ is the four vector with components (x_1, x_2, x_3, t). Similarly, $\underline{x}_o$ is a coordinate four vector. Thus G_γ^β is a second order, two point tensor. Specifically, G_γ^β corresponds to the space-like displacement at $\underline{x}$ in the β direction due to a point (vector) force at $\underline{x}_o$ in the γ direction. Since G_γ^β is space-like, then the time-like component G_4^4 is identically zero. G_γ^β plays the roles of a propagator for elastic displacement fields since solutions of (15) produce functions G_γ^β obeying the relations required for displacement field propagation in the medium from one point $(\underline{x}_o)$ to another $(\underline{x})$. In view of the meaning to be attached to the Green's function G_γ^β, only causal Green's functions are

desired, so that G_γ^β must also satisfy

$$G_\gamma^\beta(\underline{x},\underline{x}_o) = 0 \quad ; \quad x^4 < x_0^4$$

This means that G_γ^β must have the property

$$G_\gamma^\beta(x^k,x^4;x_0^k,x_0^4) = G_\gamma^\beta(x_0^k,-x_0^4;x^k,-x^4)$$

The transmission properties of the medium are expressed by the opera $^\intercal{}_\gamma$. However, the full specification of the medium, including the nature of the external and internal boundaries, is achieved by specifying boundary conditions.

Therefore complete specification of G_γ^β requires a statement of boundary conditions. For compatibility with the boundary conditions of (12) we can take:

$$\left. \begin{array}{l} \Box^\beta{}_{\alpha\gamma} G_\gamma^\beta \, \Box_\Sigma = 0 \\[2mm] \Box G_\gamma^\beta \Box_\Sigma = 0 \end{array} \right\} \tag{17}$$

or

$$\left[\left[\begin{pmatrix} \beta_{\alpha\gamma} \\ \delta_{\mu\gamma} \end{pmatrix} G_\gamma^\beta \right]\right]_\Sigma = 0 \tag{17-a}$$

At "unwelded" boundaries (solid-fluid; fluid, solid-vacuum) only the first of the relations in (17) applies in general, while the second condition is replaced by the normal velocity continuity condition,

$$[\![G^{\beta}_{\gamma,4} n_\gamma]\!]_{\Sigma} = 0 \qquad\qquad\qquad (17\text{-}b)$$

when a fluid is involved. At the exterior boundaries of the medium (usually considered as an interface with a vacuum) only the traction condition need be applied (the first condition in (17)).

The equations (15) and boundary conditions like (17) plus the causality condition, define the complete operator for G^{β}_{γ}.

The operator to be used to generate the Green's tensor $G^{\beta}_{\gamma}(\underline{x};\underline{x}_o)$ associated with the displacement field $u_{\beta}(\underline{x})$ is ideally to be chosen so that G^{β}_{γ} can be employed to propagate known initial values or boundary values of the field u_{β} through the medium, so that u_{β} can be predicted at other space-time points. Thus, G^{β}_{γ} is to be such that it acts like a simple transfer function. A necessary condition for G^{β}_{γ} to act in this manner is that it satisfy (15), plus the causality condition, plus boundary conditions accurately reflecting the properties of the medium like those of (17). However, as will be seen in the following section, when the field u_{β} is known on a particular boundary Σ_o, then it is appropriate to use homogeneous boundary conditions for G^{β}_{γ} on this boundary. (This statement applies to time-like boundaries, for initial values, as well as to space-like boundaries.) Consequently, the space-time boundary condition for Σ_o, the particular surface over which the value of u_{γ} is known or specified, is of the homogeneous form,

$$B_{\alpha\gamma} G^{\beta}_{\gamma}\Big|_{\Sigma_o} = 0,$$

with the notation here specifying that the quantity is evaluated as a limit from within the medium on one side of the surface Σ_o. The role of this choice of boundary condition on Σ_o for G_γ^β will be apparent in the following section, where it will be seen that G_γ^β will then have the characteristics of a transfer function for the displacement field u_β. However, it is both necessary and advantageous to use a causal Green's function, from the wide class generated by (15), satisfying boundary conditions other than those of the "associated" field u_β. This arises from the fact that it is usually as difficult to determine the appropriate Green's function (satisfying all the boundary conditions) as it is to obtain complete solutions for u_β itself. Thus it is usual to relax the boundary conditions on the Green's function and to obtain a Green's function that provides an approximate "transfer function" for u_β. Such procedures result in approximations for the field u_β elsewhere in the space. Aspects of this approximate technique are also discussed by Archambeau and Minster (1977), for elastodynamic relaxation source representations.

(3) Green's Tensor Integral Equations

The formally defined properties of the Green's function, in particular the differential relation (15), can be used along with the differential equation involving u_β in (12) to obtain an integral equation relating the value of the field u_β at all points in the space to known or specified values on space-time boundaries.

In particular, using Green's theorem in the four dimensional space Ω, we have for two fields with appropriate properties of continuity (Archambeau and Minster, 1977):

$$(L\underline{w},\underline{v})_\Omega = (\underline{w}, L^*\underline{v})_\Omega + \int_\Omega J_{\beta,\beta} d^4 x \qquad (18)$$

where the inner products are defined according to:

$$(L\underline{w},\underline{v})_\Omega = \int_\Omega v_\alpha L_{\alpha\gamma} w_\gamma d^4 x \qquad (18\text{-a})$$

Here $(L_{\alpha\gamma}) \equiv L$ is defined in (2). The operator L^* is the adjoint operator to L. It can be easily shown that L is a self-adjoint differential expression that is: $L \equiv L^*$.

The quantity J_β is:

$$J_\beta \equiv v_\alpha C_{\alpha\beta\gamma\delta} w_{\gamma,\delta} - w_\gamma C_{\alpha\beta\gamma\delta} v_{\gamma,\delta} \qquad (18\text{-b})$$

with $C_{\alpha\beta\gamma\delta}$ the elastic-inertial tensor defined earlier. It is evident that a formal application of Gauss' theorem to the final integral in (18) produces a "surface" integral over the boundary of Ω, involving the projection of J_β on the normal to this surface as the integrand. We note, in fact, that this

is $J_\beta n_\beta$, and that

$$J_\beta n_\beta = v_\alpha [B_{\alpha\gamma} w_\gamma] - w_\alpha [B_{\alpha\gamma} v_\gamma]$$

where $B_{\alpha\gamma}$ is the boundary operator of (12) and (17). It is this relation that explicitly displays the role of the boundary conditions in the integral equation (or solution) for u_β, which will emerge from (18).

In the important cases in which the fields, or L itself, have first or higher order discontinuities, then (18) cannot be employed in a direct, straightforward, fashion. The physical situation in which the medium has internal boundaries across which the material properties change is a particularly important example. Further, this discontinuous behavior may involve either fixed or rapidly moving boundaries, so that it may be space-time coupled.

Thus it is critical for applications that the more general discontinuous case be considered explicitly. To do so we can redefine the inner product over the space Ω, as:

$$(L\underline{w},\underline{v})_\Omega = \int_{\Omega(1)} v_\alpha L_{\alpha\gamma} w_\gamma \, d^4x + \ldots + \int_{\Omega(N)} v_\alpha L_{\alpha\gamma} w_\gamma \, d^4x \tag{19}$$

where the $\Omega^{(P)}$ are sub-regions, divided along space-time boundaries, within which all quantities in the integration are continuous. Implied here is a separation of the sub-regions by a space-time strip of width "2ϵ" along the surface of discontinuity, with the limit $\epsilon \to 0$ applied to the sum of integrals. We can, however, also write this as

$$(Lw,v)_\Omega = \int\limits_{\Omega^{(1)}\oplus \cdots \oplus\Omega^{(N)}} v_\alpha L_{\alpha\gamma} w_\gamma \, d^4x \equiv \int\limits_{\Omega\ominus\Sigma_I} v_\alpha L_{\alpha\gamma} w_\gamma \, d^4x \qquad (20)$$

if we merely take note of the fact that this integral form can always be written as a sum of integrals over the disjoint sub-regions defined by $\Omega\ominus\Sigma_I$ (or $\Omega^{(1)}\oplus\cdots\oplus\Omega^{(N)}$), where Σ_I are the <u>internal</u> surfaces of discontinuity within Ω.+ To be definite $\Omega^{(1)}$ is bounded by one of the exterior boundaries of Ω while $\Omega^{(N)}$ has, as a boundary any other exterior surface. (This "surface" may be at the origin or at infinity.) Then all the $\Omega^{(P)}, 2 \leq P \leq N$, are bounded by internal surfaces of discontinuity in the medium.

With this inner product definition in Ω we can apply the generalized Green's theorem of (18) to each subregion $\Omega^{(P)}$. It follows from (19) and (18) therefore, that:

$$(L\underline{w},\underline{v})_\Omega = (\underline{w},L^*\underline{v})_\Omega + \int\limits_{\Omega\ominus\Sigma_I} J_{\beta,\beta} \, d^4x \qquad (21)$$

Here the inner products are defined by (19) or (20) and the previously mentioned limit procedure applied to the subregion integrations is implied.

+The symbols $\oplus$ and $\ominus$ denote the set theoretic sum and difference. Thus, $\Omega\ominus\Sigma_I$ denotes the space Ω with the points of all the surfaces Σ_I deleted. For N such surfaces, this is equivalent to $\Omega^{(1)}\oplus\cdots\oplus\Omega^{(N)}$.

This result is nearly the same as (18), and indeed is the same if we always think of Ω as the set of subregions in which the fields and the differential operator are continuous. However, (21) makes this fact clear and, in addition to being more explicit than (18) in showing precisely how internal boundaries of discontinuity enter the problem, it is also rigorously derived for the general case.

Some care must clearly be used in applying Green's theorem along with Gauss' theorem to the elastodynamic problem to obtain an integral relation for the displacement field in view of the fact that internal boundaries are important. To obtain the desired relation we take w_γ in (21) to be a two point causal Green's tensor G_γ^β satisfying (15) and v_α to be a displacement field u_α satisfying the differential expression in (12). Then considering the inner product relation (21) over the coordinate space $\underline{x}_o$ (the "source coordinates, instead of the "observer" coordinates $\underline{x}$) along with differential expressions with $\underline{x}_o$ as the independent variable, one has (Archambeau and Minster, 1977)[+]:

_____

[+]Archambeau and Minster denote the integration region simply as Ω, and implicitly use the general definition of Ω as the set of subregions in which continuity holds. Here, however, we will be more explicit about this restriction on the integration and write $\Omega\Theta\Sigma_I$, with the implied limit to be taken.

$$4\pi u_\mu(\underline{x}) = \int\limits_{\Omega\Theta\Sigma_I} \rho(\underline{x}_o) \, f_\alpha(\underline{x}_o) \, G_\alpha^\mu(\underline{x};\underline{x}_o) \, d^4x^o$$

$$- \int\limits_{\Omega\Theta\Sigma_I} \frac{\partial}{\partial x_\beta^o} \left\{ G_\alpha^\mu(\underline{x};\underline{x}_o) \, \tau_{\alpha\beta}(\underline{x}_o) - u_\alpha(\underline{x}_o) \, G_{\alpha\beta}^\mu(\underline{x};\underline{x}_o) \right\} d^4x^o \tag{22}$$

The displacement field term on the left side arises from an integration of the left side of equation (21) using the generalized delta function in the differential equation for the Green's function. The first term on the right side is the same as the inner product appearing in (21), with $L_{\alpha\gamma}^* u_\gamma = L_{\alpha\gamma} u_\gamma$ replaced by its equivalent ρf_α, from equation (12). Similarly, the final integral is as given by (21), with $J_{\beta,\beta}$ being replaced by its substitutional equivalent, $J_{\beta,\beta}^\mu$, where

$$J_\beta^\mu(\underline{G},\underline{u}) = G_\alpha^\mu(\underline{x};\underline{x}_o) \, \tau_{\alpha\beta}(\underline{x}_o) - u_\alpha(\underline{x}_o) \, G_{\alpha\beta}^\mu(\underline{x};\underline{x}_o) \tag{23}$$

The Green's inertial stress tensor $G_{\alpha\beta}^\mu$ is defined analogously to the regular inertial-stress tensor $\tau_{\alpha\beta}$ (equation 4); in particular:

$$G_{\alpha\beta}^\mu(\underline{x};\underline{x}_o) = C_{\alpha\beta\gamma\delta}(\underline{x}) \, \frac{\partial G_\gamma^\mu(\underline{x};\underline{x}_o)}{\partial x_\delta} \tag{24}$$

As with $J_\beta \eta_\beta$, we note that:

$$\left.\begin{aligned} \tau_{\alpha\beta}\eta_\beta &= B_{\alpha\gamma} u_\gamma \\ G_{\alpha\beta}^\mu \eta_\beta &= B_{\alpha\gamma} G_\gamma^\mu \end{aligned}\right\} \tag{25}$$

so that the previously discussed boundary conditions appear explicitly in the integral equation (22).

The fields G_α^μ and u_α are, as noted, space-like, whereas $\tau_{\alpha\beta}$ and $G_{\alpha\beta}^\mu$ both have time-like components. Because of this mix of space-like and general four dimensional tensors in the bilinear form J_β^μ, the reduction of the final integral in (22) to a surface integral over the boundaries of Ω is most simply accomplished by separating the terms into pure space-like and time-like components. Thus, writing out the tensors in component form and separating the result into time and space-like integrals gives for the final integral in (22):

$$\int_{\Omega\Theta\Sigma_I} \frac{\partial}{\partial x_\beta^o} \left[G_\alpha^\mu \tau_{\alpha\beta} - u_\alpha \, G_{\alpha\beta}^\mu \right] d^4x^o =$$

$$\int_{\Omega\Theta\Sigma_I} \frac{\partial}{\partial x_\ell^o} \left[G_k^m \, \tau_{k\ell} - u_k G_{k\ell}^m \right] d^4x^o$$

$$-\int_{\Omega\Theta\Sigma_I} \frac{\partial}{\partial x_4^o} \left[u_k \left(\rho \, \frac{\partial G_k^m}{\partial x_4^o} \right) - G_k^m \left(\rho \, \frac{\partial u_k}{\partial x_4^o} \right) \right] d^4x^o \qquad (26)$$

The first of these can be reduced by an application of the divergence theorem to the spatial coordinates, so that:

$$\int\limits_{\Omega\Theta\Sigma_I} \frac{\partial}{\partial x_\ell^o} \left[G_k^m \tau_{k\ell} - u_k G_{k\ell}^m \right] d^4 x^o \equiv \int\limits_{\Omega^{(1)}} \frac{\partial}{\partial x_\ell^o} \left[G_k^m \tau_{k\ell} - u_k G_{k\ell}^m \right] d^4 x$$

$$+ \cdots + \int\limits_{\Omega^{(N)}} \frac{\partial}{\partial x_\ell^o} \left[G_k^m \tau_{k\ell} - u_k G_{k\ell}^m \right] d^4 x^o$$

$$= \int\limits_0^{t^+} dt_o \left[\sum_{p=1}^N \int\limits_{\partial v^{(p)}} \left[G_k^m \tau_{k\ell} - u_k G_{k\ell}^m \right] n_\ell da. \right]$$

where the sum is over the ordinary spatial (subregion) boundaries $\partial v^{(p)}$
and the integrals over time and space are separated. We can separate the
set of boundary surfaces $\partial v^{(p)}$ into two groups, that is, into those
defined to be the external boundaries of the entire medium and those that
correspond to internal boundaries. The latter, of course, always are the
common boundaries between two subregions. Because the normals $\underline{n}^{(p)}$ to the
surfaces are always directed outwardly from any subregion $v^{(p)}$, then on an
interior boundary we always have $\underline{n}^{(p)} = - \underline{n}^{(p+1)}$. Therefore the surface
integrals arising from the individual subregion integrations on common
(internal) boundaries can be combined, and we obtain:

$$\int_{\Omega\Theta\Sigma_I} \frac{\partial}{\partial x_\ell^o} \left[G_k^m \tau_{k\ell} - u_k G_{k\ell}^m \right] d^4 x^o = \int_\gamma dt_o \left[\int_{\partial\nu_E} \left\{ G_k^m \tau_{k\ell} - u_k G_{k\ell}^m \right\} n_\ell da_o \right.$$

$$\left. + \int_{\partial\nu_I} \left[\!\left[G_k^m \tau_{k\ell} - u_k G_{k\ell}^m \right]\!\right] n_\ell da_o \right] \tag{27}$$

where the double bracket notation in the second integral over all internal boundaries $\partial\nu_I$ within the medium denotes the change or jump in the quantity within the bracket across the boundary surface. That is,

$$\left[\!\left[J_\ell^m \right]\!\right] \equiv J_\ell^m(\partial\nu^{(p)}) - J_\ell^m(\partial\nu^{(p+1)})$$

for all $p = 1,2,\ldots,N$. Here J_ℓ^m denotes the tensor product in the integrand and $J_\ell^m(\partial\nu^{(p)})$ means the quantity evaluated on the internal boundary separating the $p\underline{{}^{th}}$ and $p{+}1\underline{{}^{th}}$ subregions approached (in the limit) from within the $p\underline{{}^{th}}$ subregion, while $J_\ell^m(\partial\nu^{(p+1)})$ is the same quantity evaluated in the limit by approaching the boundary from the $(p+1)$ subregion.

In (27) we have suppressed the subregion indexing (over p), so that the sum of surface integrals over each of the internal boundaries is replaced by the convention that $\partial\nu_I$ represents all the internal surfaces. The surface integration is to be taken over all the disjoint surfaces. The same convention of course, applies to $\partial\nu_E$, which represents all the external spatial surfaces of the medium.

The final integral in (26) can be written

$$\int\limits_{\Omega\Theta\Sigma_I} \frac{\partial}{\partial x_4^o}\left[u_k\left(\rho \frac{\partial G_k^m}{\partial x_4^o}\right) - G_k^m\left(\rho \frac{\partial u_k}{\partial x_4^o}\right)\right]d^4x^o \;=\;$$

$$\int\limits_0^{t^+} dt_o \int\limits_{v\Theta\partial v_I} \frac{\partial}{\partial t_o}\left[u_k\left(\rho \frac{\partial G_k^m}{\partial t_o}\right) - G_k^m\left(\rho \frac{\partial u_k}{\partial t_o}\right)\right]d^3x^o$$

where the spatial volume $v\Theta\partial v_I$ excludes the internal boundary points and implies, instead, limits from opposite sides of these surfaces approached from within the separate spatial subregions. Further, $v\Theta\partial v_I$ includes the possibility of moving boundaries, either external or internal. Therefore $v\Theta\partial v_I$ may be explicitly a function of time t_o.

Following Archambeau and Minster (1977), we observe that when $v\Theta\partial v_I$ is a function of time then the transport theorem provides, in effect, the means of interchanging the partial differentiation with respect to time and the time dependent spatial integration. Specifically, for any function (scalar, vector or tensor component) of the deformation or flow in a medium, then (Archambeau and Minster, 1977)

$$\frac{d}{dt_o} \int\limits_{v'(t_o)} F\, d^3x^o = \int\limits_{v'(t_o)} \frac{\partial F}{\partial t_o}\, d^3x^o + \int\limits_{\partial v'(t_o)} F\, U_\ell n_\ell da_o$$

where v' is any volume within the medium and U_ℓ is a component of the boundary velocity. If U_ℓ is the particle velocity of the material points then this

is the ordinary Reynolds transport theorem. The form above is a generalization of that theorem. Ordinarily, in linear elastic theory, U_ℓ is the particle velocity and the last term is neglected as being of second order. However, for rapid boundary movement -- such as for a failure surface boundary -- this term cannot be neglected.

If we apply this theorem to the last integral term in (26), taking F to be the integrand in (26), and taking care to partition the region $v\Theta\partial v_I$ along the internal boundaries, we have:

$$\int_{\Omega\Theta\Sigma_I} \frac{\partial}{\partial x_4^o} \left[u_k \left(\rho\, \frac{\partial G_k^m}{\partial x_4^o} \right) - G_k^m \left(\rho\, \frac{\partial u_k}{\partial x_4^o} \right) \right] d^4 x^o \; =$$

$$\int_0^{t+} d \left\{ \iint_{v\Theta\partial v_I} \left[u_k \left(\rho\, \frac{\partial G_k^m}{\partial t_o} \right) - G_k^m \left(\rho\, \frac{\partial u_k}{\partial t_o} \right) \right] d^3 x^o \right\}$$

$$- \int_0^{t+} dt_o \int_{\partial v_I} \left[\!\left[u_k \left(\rho\, \frac{\partial G_k^m}{\partial t_o} \right) - G_k^m \left(\rho\, \frac{\partial u_k}{\partial t_o} \right) \right]\!\right] U_\ell n_\ell da_o$$

$$- \int_0^{t+} dt_o \int_{\partial v_E} \left[u_k \left(\rho\, \frac{\partial G_k^m}{\partial t_o} \right) - G_k^m \left(\rho\, \frac{\partial u_k}{\partial t_o} \right) \right] U_\ell n_\ell da_o \qquad (28)$$

Here we have used the fact that U_ℓ is continuous across all boundaries, by definition, and that the normals from one subregion to the next change sign along their common boundaries.

The last two integrals in (28) can be combined with the integrals in (27). The first integral in (28) is to be treated as a Stieljes integral of the form

$$\int_0^{t+} df_m(t_o)$$

where $f_m(t_0)$ may have step discontinuities within (o, t^+) due to discontinuities along time-like surfaces in Ω (Archambeau, 1968, 1972; Minster, 1973; Archambeau and Minster, 1977).

Now, using (27) and (28) in (26) gives:

$$\int_{\Omega\Theta\Sigma_I} \frac{\partial}{\partial x_\beta^o} \left[G_\alpha^\mu \tau_{\alpha\beta} - u_\alpha G_{\alpha\beta}^\mu \right] d^4x^o = \int_0^{t+} dt_o \int_{\partial\nu_E} J_\beta^\mu \eta_\beta da^o$$

$$+ \int_0^{t+} dt_o \int_{\partial\nu_I} \left[\!\left[J_\beta^\mu \eta_\beta \right]\!\right] da^o - \int_0^{t+} d \left[\int_{\nu\Theta\partial\nu_I} J_4^\mu d^3x^o \right] \tag{29}$$

where we have combined surface integral terms in (27) and (28) and used the definition identities (in linearized form):

$$\tau_{\alpha\beta}n_{\beta} \equiv \left[\tau_{k\ell} - \rho \frac{\partial u_k}{\partial t_o} U_\ell \right] n_\ell$$

$$G^\mu_{\alpha\beta}n_{\beta} \equiv \left[G^m_{k\ell} - \rho \frac{\partial G^m_k}{\partial t_o} U_\ell \right] n_\ell$$

along with the fact that

$$J^\mu_\beta \equiv G^\mu_\alpha \tau_{\alpha\beta} - u_\alpha G^\mu_{\alpha\beta}$$

and

$$J^4_4 \equiv \rho \left[u_k \frac{\partial G^m_k}{\partial t_o} - G^m_k \frac{\partial u_k}{\partial t_o} \right]$$

Throughout we use the linearized representation of the generalized space-time normal n_β, which is:

$$n_\beta \equiv (n_1, n_2, n_3, - U_\ell n_\ell)$$

where n_k is the ordinary spatial normal to a surface.

Consequently, we may now write (22) in the form:

$$4\pi u_\mu(\underline{x}) = \int_{\Omega\Theta\Sigma_I} \rho(\underline{x}_o) \, f_\alpha(\underline{x}_o) \, G_\alpha^\mu(\underline{x};\underline{x}_o) \, d^4x^o$$

$$- \int_o^{t+} dt_o \int_{\partial\nu_E} J_\beta^\mu \eta_\beta da_o - \int_o^{t+} dt_o \int_{\partial\nu_I} \left[\!\left[J_\beta^\mu \eta_\beta \right]\!\right] da_o$$

$$+ \int_o^{t+} d \left[\int_{\nu\Theta\partial\nu_I} J_4^\mu \, d^3x^o \right] \tag{30}$$

Now we observe that spatial boundary conditions on the displacement field u_μ along "welded" internal boundaries are given in equation (12) as[+]:

$$\left.\left[\!\left[B_{\alpha\gamma} u_\gamma \right]\!\right]\right._{\partial\nu_I} \equiv \left.\left[\!\left[\tau_{\alpha\beta} \eta_\beta \right]\!\right]\right._{\partial\nu_I} = 0 \left.\vphantom{\begin{array}{c} \\ \\ \\ \\ \end{array}}\right\}$$

$$\left.\left[\!\left[u_\gamma \right]\!\right]\right._{\partial\nu_I} = 0 \tag{31}$$

Thus in (30), we have for the integrand, in the integral over internal surfaces:

$$\left.\left[\!\left[J_\beta^\mu \eta_\beta \right]\!\right]\right._{\partial\nu_I} = \left.\left[\!\left[G_\alpha^\mu \right]\!\right]\right._{\partial\nu_I} \tau_{\alpha\beta}\eta_\beta - u_\alpha \left.\left[\!\left[G_{\alpha\beta}^\mu \eta_\beta \right]\!\right]\right._{\partial\nu_I}$$

[+]The jump notation $\left[\!\left[\; \right]\!\right]_\Sigma$ for a surface Σ will often be written simply as $\left[\!\left[\; \right]\!\right]$, with the condition applying on any of a set of surfaces.

If then G_α^μ satisfies the same boundary conditions over the surfaces ∂v_I, namely:

$$
\left.\left[\!\left[B_{\alpha\gamma} G_\gamma^\beta \right]\!\right]\right|_{\partial v_I} = 0 \\[4pt]
\left.\left[\!\left[G_\gamma^\beta \right]\!\right]\right|_{\partial v_I} = 0
\Bigg\} \tag{32}
$$

then $J_\beta^\mu n_\beta$ itself is continuous at all internal boundaries. That is

$$
\left.\left[\!\left[J_\beta^\mu n_\beta \right]\!\right]\right|_{\partial v_I} = 0
$$

and the internal boundary integrals all vanish. Thus (31) and (32) are "natural" boundary conditions for a region with welded internal boundaries of discontinuity. Equations (32) are therefore "proper" boundary conditions for the Green's tensor, in that they are complementary to those of (31) and cause the internal surface integrals to vanish.

We note, however, that (30) has been obtained without direct use of boundary conditions on either u_γ or G_γ^β. It is therefore applicable to any set of (linear) boundary conditions for u_γ and any choice of boundary conditions for G_γ^β. It is clear, however, that a choice of "proper" boundary conditions for G_γ^β will greatly simplify (30) and that if u_γ satisfies natural boundary conditions (i.e. those naturally appearing in (30)) then additional reduction of (30) is possible.

We will denote the particular Green's tensor that satisfies the boundary conditions (32), on internal boundaries, by $H_\gamma^\beta(\underline{x};\underline{x}_o)$. Thus

$$\left[\!\!\left[B_{\gamma\alpha} H^{\beta}_{\gamma} \right]\!\!\right]_{\partial\nu_I} = 0$$

$$\left[\!\!\left[H^{\beta}_{\gamma} \right]\!\!\right]_{\partial\nu_I} = 0$$

(32-a)

and then (30) becomes:

$$4\pi u_{\mu}(\underline{x}) = \int_{\Omega\Theta\Sigma_I} \rho f_{\alpha} H^{\mu}_{\alpha} \, d^4 x^{\circ}$$

$$- \int_{0}^{t+} dt_{\circ} \int_{\partial\nu_E} J^{\mu}_{\beta} n_{\beta} \, da_{\circ} + \int_{0}^{t+} d \left[\int_{\nu\Theta\partial\nu_I} J^{\mu}_{4} d^3 x^{\circ} \right]$$

(33)

where u_{γ} satisfies the boundary conditions complementary to those of (32-a).

The external boundaries of the medium $\partial\nu_E$ may be defined rather arbitrarily in that we may chose any boundary set to be the external boundary of the medium in question and this simply defines the region over which (30) or (33) applies. In particular, we normally chose one external medium boundary to be any moving boundary so that (30) or (33) applies to the medium on one side or the other of this boundary. However, this choice is not a necessary one and if the moving boundary is not taken as an external boundary then it is to be treated as an internal boundary. However, such boundaries cannot generally be considered as welded boundaries and therefore the second condition of (31) does not apply. Instead the weaker condition insuring conservation of mass applies, that is, in linearized form:

$$\left[\!\!\left[\partial_t u_k n_k \right]\!\!\right] = 0$$

However, the first condition in (31) always applies, since it states that momentum is conserved across boundaries. Thus if one of the internal boundaries is a moving boundary, then on this particular internal boundary (call it ∂v_I^o) we have

$$\left[\!\left[J^\mu_{\;\beta} n_\beta \right]\!\right]_{\partial v_I^o} = \left[\!\left[G^\mu_{\;\alpha} \right]\!\right]_{\partial v_I^o} \tau_{\alpha\beta} n_\beta - \left[\!\left[u_\alpha G^\mu_{\;\alpha\beta} n_\beta \right]\!\right]_{\partial v_I^o}$$

Clearly, if we chose a Green's function such that it satisfies the complementary condition

$$\left[\!\left[G^\mu_{\;\alpha\beta} n_\beta \right]\!\right]_{\partial v_I^o} = 0$$

and the "welded boundary" condition:

$$\left[\!\left[G^\mu_{\;\alpha} \right]\!\right]_{\partial v_I^o} = 0$$

then

$$\left[\!\left[J^\mu_{\;\alpha\beta} n_\beta \right]\!\right]_{\partial v_I^o} = -\left[\!\left[u_\alpha \right]\!\right]_{\partial v_I^o} G^\mu_{\;\alpha\beta} n_\beta$$

In the situation in which ∂v_I^o separates materials with differences in physical properties and when the boundary is not geometrically simple, then it is impractical to generate the Green's function satisfying these boundary conditions. There are several ways around this difficulty which allow at least good approximate results. However, we need not consider them in

detail here since our ultimate goal is not to treat the general moving boundary problem but to obtain some particular results that involve fixed internal boundaries with a moving boundary, if present, as an external boundary.

Thus, in this development, we will take any moving boundary to be an external boundary. The treatment of the spatial surface integral terms, in this case, is formally the same as when all the boundaries are fixed. With this provision, we return to (33) and note that $J^{\mu}_{\beta}\eta_{\beta}$ appears in the integral term over ∂v_{E}. Thus for a solution for u_{μ} - rather than an integral equation in u_{μ} - it is necessary that we know the value of this function over the external surface. Since

$$J^{\mu}_{\beta}\eta_{\beta} = \left[H^{\mu}_{\alpha}\tau_{\alpha\beta} - u_{\alpha}H^{\mu}_{\alpha\beta} \right] \eta_{\beta}$$

and if only the generalized tractions $\tau_{\alpha\beta}\eta_{\beta}$ are known or specified; say:

$$\tau_{\alpha\beta}\eta_{\beta}\bigg|_{\partial v_{E}} = b_{\alpha} \tag{34-a}$$

then an appropriate Green's function is such that

$$H^{\mu}_{\alpha\beta}\eta_{\beta}\bigg|_{\partial v_{E}} = 0 \tag{34-b}$$

in which case the second term involving u_α vanishes. Hence, H^μ_α should

satisfy the homogeneous complementary condition[+]. Conversely, if u_α is known,

so

$$u_\alpha \Big|_{\partial v_E} = C_\alpha \tag{35-a}$$

Then the complementary homogeneous condition for the Green's function is

$$H^\mu_\alpha \Big|_{\partial v_E} = 0 \tag{35-b}$$

In these two cases we get, respectively

$$J^\mu_\beta \eta_\beta \Big|_{\partial v_E} = b_\alpha H^\mu_\alpha \tag{36}$$

and

$$J^\mu_\beta \eta_\beta \Big|_{\partial v_E} = C_\alpha H^\mu_{\alpha\beta} \eta_\beta \tag{37}$$

In these cases, assuming H^μ_α can be found satisfying both (32-a) and either

(34-b) or (35-b) as is appropriate, then (33) can (normally) be used to obtain

a solution for $u_\mu(\underline{x})$. Often it is not possible to find H^μ_α satisfying all

these conditions on all of the boundaries. In particular the conditions

(34-b) and/or (35-b) present considerable difficulties on at least parts

of the external boundary. Again there are various approximations that can

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[+]Note that on a free surface, that is an external boundary in contact with a

vacuum, then $b_\alpha = 0$, $\alpha = 1,2,3,4$. This causes $J^\mu_\beta \eta_\beta \Big|_{\partial v_E}$ to vanish when

$H^\mu_{\alpha\beta} \eta_\beta \Big|_{\partial v_E} = 0$ on such a boundary.

be used (e.g. Archambeau and Minster, 1977). However, in some applications
both u_α and $\tau_{\alpha\beta}n_\beta$ can be compatibly given as known functions on parts or all
of the external boundaries. In this case H^μ_α need not satisfy any additional
conditions, other than those of (32-a), in order to obtain a solution for u_μ.
This particular situation will be treated in the following sections since it
occurs for some particularly important problems.

The final integral in (30) or (33) corresponds to a generalized initial
value term or contribution and has been treated in this form by Archambeau (1972)
and Archambeau and Minster (1977), as well as in other equivalent forms by
Archambeau (1964,1968). In particular if the spatial integral term in brackets
is continuous in (o,t^+), then

$$\int_o^{t+} d\left[\int_{v\theta\partial v_I} J^\mu_4 d^3x^o \right] = \left[\int_{\partial\theta\partial v_I} J^\mu_4 d^3x^o \right]_o^{t+}$$

Because of the causal character of the Green's functions used, the value at
$t_o = t^+$ always is zero, so that only the value at the initial time $t_o = 0$
contributes and this involves the initial values $u_\alpha(x^o_k,o)$ and $\partial_{t_o} u_\alpha(x^o_k,o)$.
If these initial values are zero, then the integral term vanishes.

However, in general the integral can be discontinuous at a set of times
throughout the interval (o,t^+). In this case the time interval is to be
partitioned into time segments in which the integrand is continuous, just
as was done in the case of internal boundaries in the regular spatial domain.
Thus, for time boundaries at $\left\{ t_o^{(p)} \right\}_1^M$, then we have:

$$\int_{o}^{t+} d\left[\int_{v\theta\partial v_I} J_4^\mu d^3x^o\right] = \sum_{p=1}^{(L,M)} \left[\!\!\left[\int_{v\theta\partial v_I} J_4^\mu d^3x^o\right]\!\!\right]_{t_o^{(p)}} \tag{38}$$

where the range of summation over the time boundary points is to M, if $t > t_o^{(M)}$, or L, if $t < t_o^{(M)}$, where L is the index of the discrete time point in the set closest to, but less than, t. The bracket notation is the same as that previously used for jumps at boundaries, except here the boundaries are at the time points $t_o^{(p)}$.

We observe that, by definition:

$$\left[\!\!\left[\int_{v\theta\partial v_I} J_4^\mu d^3x^o\right]\!\!\right]_{t_o^{(p)}} = \lim_{\varepsilon\to 0}\left[\int_{v\theta\partial v_I} J_4^\mu d^3x^o\bigg|_{t_o=t_o^{(p)}-\varepsilon}\right.$$

$$\left.-\int_{v\theta\partial v_I} J_4^\mu d^3x^o\bigg|_{t_o=t_o^{(p)}+\varepsilon}\right] \tag{38-a}$$

Since we are considering the medium to be bounded by external boundaries that may move, then formally at least, $v\theta\partial v_I$ may be different in value at $t_o^{(p)}-\varepsilon$ and $t_o^{(p)}+\varepsilon$. However, we must define the medium in which the Green's function representation of the field u_μ holds to be that occupying the region which has not been traversed by the moving boundary. That is the volume in which the integral representation of the field $u_\mu(\underline{x})$ applies must always exclude the volume region swept out by the moving boundary in any time increment. Consequently, the change implied by (38) and (38-a) applies to a volume integral over the minimum of $v\theta\partial v_I$ at $t_o^{(p)}-\varepsilon$ and $t_o^{(p)}+\varepsilon$. Hence,

(38-a) must be written as:

$$\left[\!\!\left[\int_{\nu\Theta\partial\nu_I} J_4^\mu d^3 x^o \right]\!\!\right]_{t_o^{(p)}} = \int_{\nu_p\Theta\partial\nu_I} \left[\!\!\left[J_4^\mu \right]\!\!\right]_{t_o^{(p)}} d^3 x^o \tag{38-b}$$

where $\qquad \nu_p\Theta\partial\nu_I \equiv \min\left[\nu\Theta\partial\nu_I\right]_{t_o^P}$

Hence, the generalized initial value term of (38) is given by:

$$\int_o^{t+} d\left[\int_{\nu\Theta\partial\nu_I} J_4^\mu d^3 x^o \right] = \sum_{p=1}^{(L,M)} \int_{\nu_p\Theta\partial\nu_I} \left[\!\!\left[J_4^\mu \right]\!\!\right]_{t_o^{(p)}} d^3 x^o \tag{39}$$

Since:

$$\left[\!\!\left[J_4^\mu \right]\!\!\right]_{t_o^{(p)}} \equiv \left[\!\!\left[\rho\left\{ u_k \partial_{t_o} G_k^m - G_k^m \partial_{t_o} u_k \right\} \right]\!\!\right]_{t_o^{(p)}}$$

then it is appropriate to require that the Green's function have the continuity properties

$$\left.\begin{array}{l} \left[\!\!\left[\partial_{t_o} G_k^m \right]\!\!\right]_{t_o^{(p)}} = 0 \\[2em] \left[\!\!\left[G_k^m \right]\!\!\right]_{t_o^{(p)}} = 0 \end{array}\right\} \tag{40}$$

However, if G_k^m is simply required to be continuous in t_o along with its first derivative, for $t_o < t$, so that it is of the class C_1, then the conditions (40) are met. (Indeed such conditions hold for all times $t_o < t$, as well as at the discrete boundary times $t_o^{(p)}$, $p = 1, \ldots , M$.) Thus we require:

$$G_k^m(\underline{x},t;\underline{x}_o,t_o) \in C_1 \tag{41}$$

for $t_o < t$.

In this case

$$\left[\!\left[J_4^\mu \right]\!\right]_{t_o}(p) = \rho \left[\!\left[u_k \right]\!\right]_{t_o}(p) \; \partial_{t_o} G_k^m - \rho G_k^m \left[\!\left[\partial_{t_o} u_k \right]\!\right]_{t_o}(p) \tag{42}$$

If there are no external time varying forces applied discontinuously at the times $t_o^{(p)}$, then the momentum change $\rho \left[\!\left[\partial_{k_o} u_k \right]\!\right]$ is zero. For spontaneous processes involving boundary movement (e.g. failure or phase changes under initial stress conditions) this is the case. Then we have

$$\left[\!\left[J_4^\mu \right]\!\right]_{t_o}(p) = \rho \left[\!\left[u_k \right]\!\right]_{t_o}(p) \; \partial_{t_o} G_k^m \tag{43-a}$$

with the (time) boundary condition on u_k being:

$$\left[\!\left[\partial_{t_o} u_k \right]\!\right]_{t_o}(p) = 0 \tag{43-b}$$

Alternately we may have, when impulse forces are applied

$$\left[\!\left[J^{\mu r}_4 \right]\!\right]_{t_o}(p) = - \rho G^m_k \left[\!\left[\partial_{t_o} u_k \right]\!\right]_{t_o}(p) \tag{44-a}$$

and . the appropriate time boundary condition for the problem is

$$\left[\!\left[u_k \right]\!\right]_{t_o}(p) = 0 \tag{44-b}$$

By far the most important case in applications is that given in (43-a) and (43-b). Specifically, Archambeau (1968,1972) and Archambeau and Minster (1977) consider failure in a prestressed medium and show that (43-a) and (43-b) can be used to describe the radiation field arising from the failure process. In this case (43-a) applies and $\left[\!\left[u_k \right]\!\right]_{t_o}(p)$ corresponds to the change in the equilibrium field due to the creation of a failure boundary (or its incremental growth) so that in this application:

$$\left[\!\left[J^{\mu}_4 \right]\!\right]_{t_o}(p) = \rho \left[\!\left[u^*_k \right]\!\right]_{t_o}(p) \; \partial_{t_o} G^m_k$$

where u^*_k is used to denote the equilibrium field explicitly.

Equation (39) expresses the contribution to the field $u_\mu(\underline{x})$ arising from the discontinuous behavior of the spatial volume integral of J^{μ}_4 at a discrete set of times $\left\{ t_o^{(p)} \right\}_1^M$. We may generalize this result to include a representation in which the set $\left\{ t_o^{(p)} \right\}_1^M$ becomes partly or totally a continuous distribution. The result is a straightforward generalization

of (39) which has been shown by Archambeau (1968, 1972) and Archambeau
and Minster (1977), in the somewhat more restricted context of a growing
failure surface problem, to be of the form:

$$\int_0^{t+} d\left[\int_{v\theta\partial v_I} J_4^\mu d^3x^o\right] = \int_0^{\min(t,t_o^{(M)})} dt_o \int_{v\theta\partial v_I}\left(\delta J_4^\mu/\delta t_o\right) d^3x_o \tag{45}$$

where $t_o^{(M)}$ is the upper limit point of the continuous and/or discrete set
of discontinuity points and δJ_4^μ represents the incremental change or
variation of J_4^μ over a time increment δt_o. In (45) all quantities are to
be measured relative to the initial state. (See Archambeau and Minster, 1977
for other choices for the reference state.) Here δJ_4^μ is equivalent to
$[\![J_4^\mu]\!]$ in (39), while the time integral over t_o replaces the sum over the
index p. Formally, the integral is the limit of the summation as the time
spacing, δt, between successive $t_o^{(p)}$ is allowed to become infinitesimal.

To explicitly display the discrete discontinuous case, it is only
necessary to note that discrete discontinuities in J_4^μ at times $t_o^{(p)}$ correspond
to step function discontinuities of magnitude $[\![J_4^\mu]\!]_{t_o^{(p)}}$, at $t_o = t_o^{(p)}$. Then

$$\left(\delta J_4/\delta t_o\right) = [\![J_4^\mu]\!] \delta\left(t_o - t_o^{(p)}\right)$$

with $\delta(t_o - t_o^{(p)})$ the Dirac delta function. It is clear that use of this in
(45) gives the discrete result (39). Hence (45) is a general form of the
result including both discrete and continuous distributions of time discontinuities.

For the cases of greatest interest, namely when $\partial_{t_o} u_k$ is continuous and G_k^m, and its first derivative in t_o are continuous, then the integrand in (45) has the explicit form (e.g. Archambeau and Minster, 1977)

$$\left(\delta J_4^\mu / \delta t_o\right) = \left[\rho\, \partial_{t_o} u_k^*\right] \partial_{t_o} G_k^m \tag{46}$$

where $u_k^*(\underline{x}_o, t_o)$ is the equilibrium field value at the source time t_o. The factor $\delta u_k^*/\delta t_o$ has been written as a partial derivative with respect to t_o since the meaning is the same as the variation ratio. The general case is

$$\left[\delta J_4^\mu / \delta t_o\right] = \rho \left(\delta u_k / \delta t_o\right)\partial_{t_o} G_k^m - \rho G_k^m \left(\delta v_k / \delta t_o\right)$$

where $v_k \equiv \partial_{t_o} u_k$.

To collect and summarize the most useful results of this section, we have shown that in general, the integral Green's tensor representation of $u_\mu(\underline{x})$ is:

$$4\pi u_\mu(\underline{x}) = \int_o^{t+} dt_o \int_{v\theta\partial v_I} \rho f_\alpha\, G_\alpha^\mu d^3x^o$$

$$- \int_o^{t+} dt_o \int_{\partial v_E} J_\beta^\mu \eta_\beta da_o - \int_o^{t+} dt_o \int_{\partial v_I} \left[\!\left[J_\beta^\mu \eta_\beta \right]\!\right] da_o$$

$$+ \int_o^{\tau} dt_o \int_{v\theta\partial v_I} \left(\delta J_4^\mu / \delta t_o\right) d^3x_o \tag{47}$$

where the result (45) has been used in (30) and $\tau = \min(t, t_o^M)$. The Green's tensor used to obtain (47) is the causal solution of $L_{\alpha\gamma} G_\gamma^\beta = \Delta_\alpha^\beta$ satisfying the condition that G_γ^β and its first time derivative with respect to t_o be continuous for $t_o < t$. When the medium internal boundaries are fixed and welded and any moving boundaries are external boundaries, then $u_\mu(\underline{x})$ satisfies the boundary condition

$$\left[\!\left[B_{\alpha\gamma} u_\gamma \right]\!\right]_{\partial v_I} \equiv \left[\!\left[\tau_{\alpha\beta} n_\beta \right]\!\right]_{\partial v} = 0 \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\}$$

$$\left[\!\left[u_\gamma \right]\!\right]_{\partial v_I} = 0$$

$\qquad$ (48)

on all internal boundaries. Then,

$$\left[\!\left[J_\beta^\mu n_\beta \right]\!\right]_{\partial V_I} = \left[\!\left[G_\alpha^\mu \right]\!\right]_{\partial v_I} \tau_{\alpha\beta} n_\beta - u_\alpha \left[\!\left[G_{\alpha\beta}^\mu n_\beta \right]\!\right]_{\partial v_I}$$

$\qquad$ (48-a)

Further for spontaneous processes giving rise to generalized initial values, such as failure processes involving moving (or growing) boundaries in an initially stressed medium, then

$$\left(\delta J_4^\mu / \delta t \right) = \left[\rho \, \partial_{t_o} u_k^* \right] \partial_{t_o} G_k^m$$

with u_k^* the equilibrium field for the medium within the external boundaries. Here $\partial_{t_o} u_k^*$ is the "source time" derivative of u_k^* and is the source term for the radiation field arising from the process. In principle $u_k^*(\underline{x}_o, t_o)$ can be obtained

independently from (47) (e.g. Archambeau, 1968) so that the integrand of the

final term in (47) can be considered as specified. Likewise the applied

force field $f_\alpha(\underline{x}_o)$ can be considered as a known field. Thus these contributions

to u_μ can be obtained directly, assuming that G_α^μ, satisfying the conditions

just stated, is obtained. In view of (48-a) however, the surface integral over

∂v_I, the internal boundaries, contains u_α so that (47), as it stands, is

an integral equation for u_α. Further the surface integral over ∂v_E, the

external boundaries, may contain u_α terms as unknowns.

If we use the particular Green's tensor $H_\alpha^\mu(\underline{x};\underline{x}_o)$ which satisfies the

internal boundary conditions complementary to those for u_α in (48), namely

$$\left[\!\left[B_{\alpha\gamma} H_\gamma^\mu \right]\!\right]_{\partial v_I} \equiv \left[\!\left[H_{\alpha\beta}^\mu n_\beta \right]\!\right]_{\partial v_I} = 0 \tag{49}$$

$$\left[\!\left[H_\gamma^\mu \right]\!\right]_{\partial v_I} = 0$$

then with H_α^μ used for G_α^μ in (48-a) we have

$$\left[\!\left[J_\beta^\mu n_\beta \right]\!\right]_{\partial v_I} \equiv 0$$

and the integral over the internal boundaries ∂v_I in (47) vanishes.

Further, if the external boundary of the medium is partly or wholly a

free surface, then the total external boundary may be denoted as

$$\partial v_E = \partial v_E^{(0)} + \partial v_E^{(1)} \tag{50}$$

with $\partial v_E^{(0)}$ denoting a free surface and $\partial v_E^{(1)}$ the remaining part of the external boundary. On $\partial v_E^{(0)}$ the boundary condition for u_μ is:

$$\left[\!\left[B_{\alpha\gamma} u_\gamma \right]\!\right]_{\partial v_E^{(0)}} = \left[\!\left[\tau_{\alpha\beta} n_\beta \right]\!\right]_{\partial v_E^{(0)}} = 0 \tag{51}$$

The complementary condition for the Green's tensor is:

$$\left[\!\left[B_{\alpha\gamma} H_\gamma^\mu \right]\!\right]_{\partial v_E^{(0)}} = \left[\!\left[H_{\alpha\beta}^\mu n_\beta \right]\!\right]_{\partial v_E^{(0)}} = 0 \tag{52}$$

In this case, using H_γ^μ satisfying (49) and (52) with u_γ satisfying (48) and (51)

$$\left. J_\beta^\mu n_\beta \right|_{\partial v_E^{(0)}} \equiv 0$$

Thus, with the Green's tensor H_γ^μ, the integral relation for $u_\mu(\underline{x})$ is:

$$4\pi u_\mu(\underline{x}) = \int_0^{t+} dt_o \int_{v\theta\partial v_I} \rho f_\alpha H_\alpha^\mu \, d^3 x^o$$

$$- \int_0^{t+} dt_o \int_{\partial v_E^{(1)}} J_\beta^\mu n_\beta da_o \tag{53}$$

$$+ \int_0^\tau dt_o \int_{v\theta\partial v_I} \left[\rho \, \partial_{t_o} u_\alpha^* \right] \partial_{t_o} H_\alpha^\mu \, d^3 x_o$$

where

$$\left[J^\mu_\beta \eta_\beta = H^\mu_\alpha \tau_{\alpha\beta} - u_\alpha H^{\ \mu}_{\alpha\beta} \right] \eta_\beta$$

If values of the fields u_α or $\tau_{\alpha\beta}\eta_\beta$ are known or specified on $\partial v_E^{(1)}$ then complementary homogeneous conditions on H^μ_α or $H^\mu_{\alpha\beta}\eta_\beta$ reduces $J^\mu_\beta\eta_\beta$ on $\partial v_E^{(1)}$ to a known function, not involving the unknown u_α or its derivatives, and (53) is a solution for $u_\mu(\underline{x})$. That is, in this case, H^μ_α in combination with its space and time derivative, act as transfer functions for the field specified on space and time boundaries, such that the field is propagated to other spatial points at later times.

Alternately, if compatible values of both u_α and $\tau_{\alpha\beta}\eta_\beta$ are known or can be specified on $\partial v_E^{(1)}$, then H^μ_α need not satisfy any additional condition in order that (53) provide a solution for $u_\mu(x)$. That is $J^\mu_\beta\eta_\beta$, in this case, is a known function on $\partial v_E^{(1)}$ when the Green's tensor, H^μ_α, satisfying (only) (49) and (52) is used.

In the following sections this latter case will be demonstrated to occur in several important applications. In order to use (53), however, H^μ_α must be specified. The generation of H^μ_α for media of the greatest interest in elastodynamic theory applications to Geophysics are layered elastic spheres and half spaces. The form of H^μ_α in such media will therefore be considered in following sections.

(4) Green's Tensor Integral Representations in the Frequency Domain

For this development to be pursued, it is necessary to transform all the equations and fields from the time domain to the frequency domain. This was the approach used by Archambeau (1968) from the onset in treating the particular problem of a growing rupture zone in a stressed medium, this being a particular case of the general problem treated here. We can however use the time domain integral representation for u_α given in the previous section, to obtain the equivalent frequency domain result.

In particular, we define the Fourier transform operation with respect to time t by

$$\tilde{u}_\alpha \equiv \oint_t \left\{ u_\alpha \right\} \equiv \int_{-\infty}^{+\infty} u_\alpha \, e^{-i\omega t} dt \tag{54}$$

Here $\oint_t$ denotes the operator and $\tilde{u}_\alpha$ the transformed function. The inverse operator $\oint_\omega^{-1}$ is defined by

$$u_\alpha = \oint_\omega^{-1} \left\{ \tilde{u}_\alpha \right\} = \int_{-\infty}^{+\infty} \tilde{u}_\alpha \, e^{i\omega t} df$$

where $\omega = 2\pi f$ is the transform variable and here corresponds to angular frequency with f the frequency.

The convolution of two (tensor) functions is then given by

$$\oint_\omega^{-1} \left\{ \tilde{u}_\alpha \tilde{v}_\beta \right\} = \int_{-\infty}^{+\infty} u_\alpha(t_o) \, v_\beta(t-t_o) dt_o$$

and if u_α is such that $u_\alpha(t_o) = 0$ for $t_o \leq 0$ while $v_\alpha(t_o) = 0$ for $t_o \geq t^+ = t + \varepsilon$, then this becomes:

$$\delta_\omega^{-1} \left\{ \tilde{u}_\alpha \tilde{v}_\beta \right\} = \int_0^{t+} u_\alpha(t_o) v_\beta(t-t_o) dt_o$$

In this case

$$\delta_t \, \delta_\omega^{-1} \left\{ \tilde{u}_\alpha \tilde{v}_\beta \right\} \equiv \tilde{u}_\alpha \tilde{v}_\beta = \delta_t \left\{ \int_0^{t+} u_\alpha(t_o) v_\beta(t-t_o) dt_o \right\}$$

Convolution relations of this type can be seen to appear in the Green's integral relations for the elastic field $u_\alpha(\underline{x})$, for example in (53). Thus since v_β in the above relations plays the role of a transfer function, then it is clear that the various integral kernels involving Green's functions play the same role. This will become even more evident in the following development.

Focusing our development on the most useful of the Green's integral relations, we consider the Fourier transform of $u_\alpha(\underline{x})$ in (53) with respect to the (observers) time variable t. In order to carry out this operation in the most routine way, we first observe that the two point Green's tensor $H_\alpha^\mu(x^k, t; x_o^k, t_o)$ is causal, so that

$$H_\alpha^\mu(\underline{x}; \underline{x}_o) = 0, \text{ for } t_o > t$$

This allows us to extend the integration range in the first and second integral terms from (o, t^+) to (o, ∞) without chaning the value of these integrals,

since H_α^μ vanishes identically when $t_o \geq t^+$. Next we note that:

$$u_\alpha(t_o) = 0 \text{ for } t_o < 0$$

by supposition (i.e. the field effects start at $t_o = 0$). This allows us to integrate over the "source time" range $(-\infty, +\infty)$ in the first two integrals since these integrals vanish in the range $(-\infty, o)$ because of the latter condition. Finally, for the last integral involving the "generalized initial values", we can take the time changes in the equilibrium field to be zero in the range outside the interval (o, t_o^M) since such changes only occur in this interval by our earlier definitions. Thus

$$\partial_{t_o} u_\alpha^*(t_o) = 0; \quad t_o < 0, \ t_o > t_o^M$$

This, in combination with $H_\alpha^\mu = 0$ for $t_o > t$, shows that the integration range for the final integral can be extended to $(-\infty, +\infty)$ without changing the value of the integral as well. Thus (53) can be written as

$$4\pi u_\mu = \int_{-\infty}^{\infty} dt_o \int_{v\theta\partial v_I} \rho f_\alpha \ H_\alpha^\mu d^3 x_o - \int_{-\infty}^{+\infty} dt_o \int_{\partial v_E^{(1)}} J_\beta^\mu \eta_\beta \ da_o$$

$$+ \int_{-\infty}^{+\infty} dt_o \int_{v\theta\partial v_I} \left[\rho \ \partial_{t_o} u_\alpha^* \right] \partial_{t_o} H_\alpha^\mu d^3 x_o$$

$$\tag{55}$$

Finally we observe that the Green's tensors are functions which depend only on the time difference $t-t_o$. This follows from the fact that they are required to be causal and because of the self-adjointness of the generating differential operator $L_{\alpha\gamma}$. Thus we have that:

$$H^\mu_\alpha \equiv H^\mu_\alpha(x^k, x^k_o, t-t_o).$$

Now operating on equation (55) with δ_t, where we note that this operator commutes with all the integral operators on the right side of (55), we get:

$$4\pi\bar{u}_\mu = \int_{-\infty}^{+\infty} e^{-i\omega t_o} dt_o \int_{\nu\Theta\partial\nu_I} \rho f_\alpha \, \tilde{H}^\mu_\alpha d^3x_o$$

$$- \int_{-\infty}^{+\infty} e^{-i\omega t_o} dt_o \int_{\partial\nu_E} \left[\tilde{H}^\mu_\alpha \tau_{\alpha\beta} - u_\alpha \, \tilde{H}^\mu_{\alpha\beta} \right] n_\beta da_o \tag{56}$$

$$+ i\omega \int_{-\infty}^{+\infty} e^{-i\omega t_o} dt_o \int_{\nu\Theta\partial\nu_I} \left[\rho \partial_{t_o} u^*_\alpha \right] \tilde{H}^\mu_\alpha d^3x_o$$

Here we have used the fact that only H^μ_α and $H^\mu_{\alpha\beta}$ depend on t on the right side of (55) and that

$$\delta_t \left\{ H^\mu_\alpha(x^k, x^k_o, t-t_o) \right\} = e^{-i\omega t_o} \delta_t \left\{ H^\mu_\alpha(x^k, x^k_o, t) \right\} = e^{-i\omega t_o} \tilde{H}^\mu_\alpha$$

If the medium boundary $\partial v^{(1)}$ is not stationary, then $v\theta\partial v_I$ and $\partial v_E^{(1)}$ will be functions of t_o, so that the space integrals in (56) must be evaluated first and then the integral over t_o may be obtained. Thus the time integral over t_o which is itself in the form of a Fourier transform operator, does not commute with spatial integral when there is boundary movement. (Of course approximations can be obtained by neglecting the volume or surface area changes and treating the limits of the spatial integrals as fixed in time, at least to first order.)

If all the boundaries of the problem are fixed (in the sense of the usual elastic theory approximation in which boundary movement with the particles is considered as a fixed boundary), then the integration with respect to t_o and the space integrals can be interchanged. We have then:

$$4\pi\tilde{u}_\mu = \int_{v\theta\partial v_I} \rho\tilde{f}_\alpha \ \tilde{H}^\mu_\alpha d^3x_o$$

$$- \int_{\partial v_E^{(1)}} \left[\tilde{H}^\mu_\alpha \tilde{\tau}_{\alpha\beta} - \tilde{u}_\alpha \ \tilde{H}^\mu_{\alpha\beta} \right] \eta_\beta da_o \tag{57}$$

$$- \omega^2 \int_{v\theta\partial v_I} \rho\overset{*}{\tilde{u}}_\alpha \ \tilde{H}^\mu_\alpha d^3x_o$$

Here the boundaries are to be considered as fixed after the time $t_o = 0$. However, this representation still admits the case of an instantaneously created boundary at $t_o = 0$. In this case $\overset{*}{u}_\alpha(x^k_o, t_o)$ is a step function at $t_o = 0$ whose transform $\overset{*}{\tilde{u}}_\alpha$ is proportional to $1/\omega$. This problem was treated

by Archambeau (1972) using a representation form of essentially the type (57), with only the last term retained as significant.

It is also worth pointing out that since the boundaries are assumed fixed here, then $U_\ell = 0$ in (57). In this case all the tensors are purely space-like; for example $n_\beta = (n_1, n_2, n_3, 0)$. This means that all the Greek indices in (57) may be replaced by three term indices, denoted by the latin forms, ℓ, k,.., etc.

The general structure of (57), when viewed in terms of the convolution definitions previously stated, is that of a sum of convolution operations. In particular, all the terms in (57) involving forces, that is $\rho \tilde{f}_\alpha$, $\tilde{\tau}_{\alpha\beta} n_\beta$ $-\rho\omega^2 \tilde{u}_\alpha^*$, have $\tilde{H}_\alpha^\mu$ as a (time-like) transfer function. On the other hand, the term involving the boundary displacement $\tilde{u}_\alpha$ on $\partial v_E^{(1)}$, has $\tilde{H}_{\alpha\beta}^\mu n_\beta$ as a transfer function. Moreover, the Green's function will be dependent on the spatial coordinates x^k and x_o^k through the absolute difference $x^k - x_o^k$. That is, its functional dependence can be expressed as

$$H_\alpha^\mu \equiv H_\alpha^\mu \left(|x^k - x_o^k|, \; t - t_o \right)$$

This follows from the causal condition imposed on H_α^μ, coupled with the character of the generating equation $L_{\alpha\gamma} H_\gamma^\beta = \Delta_\alpha^\beta$. These of course are the same conditions leading to the time difference dependence used earlier and lead to the usual statement of reciprocity for the Green's function.

Therefore with H_α^μ of the functional form described, it is clear that the spatial integrations in (57) are also in the form of (three dimensional space) convolutions. Therefore H_α^μ and $H_{\alpha\beta}^\mu n_\beta$ are seen, from (57), to have the character of space-time transfer functions or propagators.

The form of (57) is considerably simpler than that of the more general result (56), which admits of the treatment of moving boundaries. However, the structure of the two integral representations is essentially the same. In particular, H_α^μ and $H_{\alpha\beta}^\mu n_\beta$ are the propagators for the fields. Since $H_{\alpha\beta}^\mu$ is related to H_α^μ within the integrands in (56) and (57) by:

$$H_{\alpha\beta}^\mu (\underline{x};\underline{x}_o) = C_{\alpha\beta\gamma\delta}(\underline{x}_o) \frac{\partial H_\gamma^\mu}{\partial x_o^\delta}$$

Then the time transforms with respect to $t = x^4$ are related by

$$\tilde{H}_{\alpha\beta}^\mu \equiv \mathcal{E}_t \left\{ \tilde{H}_{\alpha\beta}^\mu \right\} = C_{\alpha\beta\gamma\delta}(\underline{x}_o) \frac{\partial \tilde{H}_\gamma^\mu}{\partial x_o^\delta} \qquad (58)$$

Therefore knowledge of $\tilde{H}_\gamma^\mu$, the transform with respect to the receiver time coordinate, is sufficient to specify the tensor $\tilde{H}_\gamma^\mu$ appearing as a transfer function in both (56) and (57).

Thus, it has now been shown that knowledge of $\tilde{H}_\gamma^\mu$ is all that is required in order to use (56) and (57) to solve a large class of elastodynamic problems, including problems involving moving boundaries, as well as classical space-time boundary value problems. Therefore solutions to a number of important elastodynamic problems require the solution of the receiver time transformed equation corresponding to:

$$L_{\alpha\gamma} H_\gamma^\beta (\underline{x};\underline{x}_o) = \Delta_\alpha^\beta (\underline{x};\underline{x}_o) \qquad (59)$$

with boundary conditions, from equations (49) and (52),

$$\left[\!\left[H^{\mu}_{\alpha\beta}{}^{n}{}_{\beta} \right]\!\right]_{\partial v_{I}} = 0$$

$$\left[\!\left[H^{\mu}_{\gamma} \right]\!\right]_{\partial v_{I}} = 0 \tag{60}$$

$$\left[H^{\mu}_{\alpha\beta}{}^{n}{}_{\beta} \right]_{\partial v_{E}^{(0)}} = 0$$

By our previous definitions of the boundaries ∂v_{I} and $\partial v_{E}^{(0)}$, These are all fixed boundaries while only $\partial v_{E}^{(1)}$ may be a moving boundary. Thus all the tensors in (60) are space like (i.e., $U_{\ell} \equiv 0$ on these boundaries) and so these conditions may be written in terms of space-like components above. We have:

$$\left[\!\left[H^{m}_{k\ell}{}^{n}{}_{\ell} \right]\!\right]_{\partial v_{I}} = 0$$

$$\left[\!\left[H^{m}_{k} \right]\!\right]_{\partial v_{I}} = 0 \tag{60-a}$$

$$\left[H^{m}_{k\ell}{}^{n}{}_{\ell} \right]_{\partial v_{E}^{(0)}} = 0$$

Taking the transform $\oint_{t}$ of these equations for H^{β}_{α} gives:

$$L_{\ell k} \tilde{H}^m_k + \rho \omega^2 \tilde{H}^m_\ell = - 4\pi \delta^m_\ell \delta (x^k - x^k_o) \tag{61}$$

where $\tilde{H}^m_k$ is a function of the ordinary space coordinates alone, and where

$$L_{\ell k} = \frac{\partial}{\partial x_j} (C_{\ell j k m} \frac{\partial}{\partial x_m})$$

is the regular elastic operator. The boundary conditions become:

$$\left[\!\left[\tilde{H}^m_{k\ell} n_\ell \right]\!\right]_{\partial \nu_I} = 0$$

$$\left[\!\left[\tilde{H}^m_k \right]\!\right]_{\partial \nu_I} = 0 \tag{62}$$

$$\left[\tilde{H}^m_{k\ell} n_\ell \right]_{\partial \nu^{(0)}_E} = 0$$

where

$$\tilde{H}^m_k = C_{k\ell ij} \frac{\partial \tilde{H}^m_i}{\partial x_j}$$

The boundary conditions may be written in a compact matrix form, analogous to that expressed earlier in the equation (14), as:

$$\left[\!\left[\begin{pmatrix} B_{\ell k} \\ \delta_{nk} \end{pmatrix} \tilde{H}^m_k \right]\!\right]_{\partial \nu_I} = 0 \tag{62-a}$$

$$\left[\mathcal{B}_{\ell k} \bar{H}^m_k \right]_{\partial \nu_E (0)} = 0 \tag{62-b}$$

where $\mathcal{B}_{\ell k}$ is the "space boundary operator"

$$\mathcal{B}_{\ell k} = n_j \left[C_{\ell j k i} \frac{\partial}{\partial x_i} \right]$$

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Two major advances have been made in the research on synthesis of long-period S waves at distances less than 40 deg: the incorporation of inelastic attenuation and the development of a code to include a representation of the source function. The large effects on the waveforms of rapid variations of velocity in the upper mantle can be calculated and separated from the effects of anelasticity.

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20.

Examples of synthetic seismograms are included to illustrate the effects of attenuation on the displacement waveforms, as well as the effects of the response of a typical long-period seismograph system.

The application of Green's Function techniques to elastodynamics has led to methods for treating a variety of problems in wave propagation and earthquake source representations. The complete theory is given in this report. In particular, wave propagation in a realistic, layered earth from generalized nonlinear sources is treated specifically, the dynamic field due to stress relaxation around a geometrically general, growing inclusion (an earthquake source) in a spatially heterogeneous initial stress field is considered and computations based on the theoretical results obtained are currently being carried out.

There is evidence that specific anelastic attenuation, expressed as Q, is frequency dependent. Q models of the Earth based on free oscillations, surface waves and body waves show that in the broad frequency band covered by the input data, Q increases with frequency. Frequency-dependent Q is modelled by a relaxation process with a range of relaxation times. An investigation of the relaxation time characterizing the high frequency corner of the absorptionband was carried out using the data from 21 shallow earthquakes, 1 at intermediate depth, and 4 deep ones.

The criteria used was that the Q-corrected spectrum show decay at high frequencies at a rate in the range f^{-2} to f^{-3}. The depth dependence of the resulting relaxation times and corresponding values of T/Q was examined. mixed effect of depth and frequency dependence of P-wave attenuation was found. The important conclusions is that the P and S attenuation data can only be reconciled by including a bulk loss mechanism, in addition to a shear loss mechanism. Although the results are not unique, this suggests that the bulk loss mechanism is operative in the upper mantle, perhaps within the asthenosphere.